PERTURBATION ANALYSIS OF NONLINEAR REACTOR KINETICS WITH DOPPLER FEEDBACK

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THESIS

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ABSTRACT

The analysis of nonlinear reactor kinetics with Doppler feedback is developed by the use of the Poincaré-Lindstedt perturbation technique. The two types of temperature dependent reactivity coefficients to be investigated are

$$\frac{dk}{dT} \propto \frac{1}{T} \quad \text{and} \quad \frac{dk}{dT} \propto \frac{1}{T^{3/2}}.$$

Each of these reactivity models are analyzed for the reactor with positive and negative coefficients of reactivity and infinite slab and finite cylindrical geometries. The Poincaré-Lindstedt perturbation solution is constructed in the form of an expansion about the unperturbed state of the reactor. Each of the perturbed equilibrium solutions is subsequently analyzed for stability.

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TABLE OF SYMBOLS

a	temperature coefficient
C_i	i^{th} group precursor concentration
D	diffusion coefficient
E	energy
g	fast nonleakage probability
j	free index of odd integers
m	free index of odd intergers
N	number of precursor groups
n	index describing mode number
p	resonance escape probability
P_t	thermal nonleakage probability
k	multiplication factor
k_∞	multiplication factor of an infinite reactor
T	temperature
t	time
v	neutron velocity
x,r,z	dimensionless spacial coordinate
α	stability parameter
β_i	fraction of the fission neutrons from the i^{th} precursor group
ϵ	perturbation parameter
η	initial deviation of the equilibrium state

λ	system parameter (material buckling)
λ_0	unperturbed material buckling
λ_i	decay constant for the i^{th} precursor
Σ	macroscopic neutron cross section
σ	microscopic cross section
τ	dimensionless time
φ	neutron flux distribution
φ_0	initial flux distribution
Ψ_m, Γ_m	shape function
$(^\circ)$	partial differentiation with respect to ϵ

I. THE NONLINEAR TEMPERATURE DEPENDENT REACTOR

A. GOVERNING EQUATIONS

In a homogeneous, one-velocity, bare reactor the neutron population is described by the equation (1)

$$\frac{1}{\nu}\frac{\partial \varphi(\underline{x},t)}{\partial t} = D\nabla^2\varphi(\underline{x},t) - \Sigma_a \varphi(\underline{x},t) \qquad (1)$$

$$k(1-\beta)\Sigma_a\varphi(\underline{x},t) + p\sum_{i}^{N}\lambda_i C_i(\underline{x},t) \quad .$$

The first term Eq. (1) represents the time rate of change of the neutron flux within a differential control of volume. This term is balanced by the net effect of the diffusion of neutrons, the absorption of neutrons, and the production of both prompt and delayed neutrons within the volume. The concentration of the precursors is described by [1]

$$\frac{\partial C_i(\underline{x},t)}{\partial t} = -\lambda_i C_i(\underline{x},t) + k\frac{\Sigma_a \beta_i}{p}\varphi(\underline{x},t) \qquad (2)$$

$$i = 1, 2, 3, \cdots, N \quad .$$

The solution of these equations with temperature dependent parameters represents the goal of this work.

The temporal and spacial coordinates may be nondimensionalized by introducing the definitions

$$\chi = \sqrt{\Sigma_a/D}\;\underline{x} \quad \text{and} \quad \tau = \nu\Sigma_a t \quad .$$

Equations (1) and (2) become the following

$$\frac{\partial \varphi}{\partial \tau} = \nabla^2 \varphi - \varphi + k(1-\beta)\varphi + \frac{p}{\Sigma_a} \sum_i^N \lambda_i C_i \quad (3)$$

$$\frac{\partial C_i}{\partial \tau} = -\frac{\lambda_i}{\nu \Sigma_a} C_i + k \frac{\beta_i}{\nu p} \varphi \quad . \quad (4)$$

where the precursor concentration and the flux must have a zero value on the boundaries of the reactor. The solution is desired for the infinite slab reactor and the finite cylindrical reactor. The first of these cases is the slab which may be envisioned as a reactor which is infinite in two dimensions and bounded by the coordinates +1 and -1 in the third dimension. This simple geometry will easily lend itself to analysis, but permits the observance of mathematical rigor and the physical interpretation of the results. The boundary conditions require that the flux is zero at the coordinates ± 1. The second case is that described by a finite cylinder which has boundaries on the longitudinal axis at ± 1 and at a nondimensional radial distance of r_o. Therefore, the flux is required to be zero at the extremes of the longitudinal coordinate, $Z = \pm 1$, and at the radius $r = r_o$.

B. DOPPLER EFFECT

The lumped parameter k is generally referred to as the neutron multiplication factor. In elementary reactor analysis the multiplication factor is evaluated as a constant given by the product [1]

$$k = k_\infty P_T \eta$$

However, in reactors that operate at high temperatures the multiplication factor becomes temperature dependent as will subsequently be shown.

One of the major considerations to be made when accounting for the temperature dependence of the multiplication factor is that described by the nuclear Doppler effect. Physically, this phenomenon results from the temperature broadening of the capture cross-sections of the fuel nucleii. Thus the Doppler effect may be envisioned by a temperature dependent competition for neutrons by capture and fission processes. The quantitative description of the Doppler effect is achieved by making the following assumptions:

a. The fuel density as a function of velocity is given by a Maxwellian distribution function.

b. The capture cross-sections near an isolated resonance is given by the Breit-Wigner formula.

It may be shown that the average radiative cross-section, σ_γ, and scattering cross-section, σ_s, are described as a function of energy and temperature given by [1]

$$\sigma_\gamma(E,T) = \frac{\sigma_1 \Gamma_\gamma}{\Gamma} \psi(\xi, \hat{x})$$

and

$$\sigma_s(E,T) = \frac{\sigma_1 \Gamma_\gamma}{\Gamma} \psi(\xi,\hat{x}) + \frac{\sigma_1 R}{\lambda} \chi(\xi,\hat{x}) + 4\pi R^2$$

where $\psi(\xi, \hat{x})$ and $\chi(\xi, \hat{x})$ are tabulated functions of temperature,

$$\hat{x} = \frac{2}{\Gamma}(E - E_1)$$

$$\xi = \frac{\Gamma}{\sqrt{\frac{4E_1 kT}{A}}} .$$

The effect of σ_γ and σ_s is most conveniently incorporated into one quantity by introducing the effective resonance integral, I, defined by [2]

$$I = \int (\sigma_a)_{EFFECTIVE} \frac{dE}{E} = \int \frac{\sigma_\gamma \sigma_M}{\sigma_a + \sigma_s + \sigma_M} \frac{dE}{E}$$

where σ_M is the additional scattering cross-section per fuel atom. It is usually more convenient to analyze the temperature dependence through the J function which is related to the resonance integral as follows [1]

$$I = \frac{\sigma_M \Gamma_\gamma}{E_r} J(\xi, \beta)$$

where

$$\beta = \frac{N_M \sigma_{sM} \Gamma}{N_A \sigma_1 \Gamma_\gamma}$$

and the subscript M refers to the moderator.

If attention is returned to the concept of the multiplication factor, its dependence upon temperature may be investigated [1]

Since

$$k \propto p$$

and

$$p \propto \exp(I) ,$$

the Doppler effect on k is studied by considering the temperature effect on p. However, the preceding analysis showed that

$$I = \frac{\sigma_m \Gamma_\gamma}{E_r} J(\xi,\beta)$$

and it may be simply concluded that

$$f_k \propto \exp\left[J(\xi,\beta)\right] .$$

If the derivative with respect to temperature is taken, it is obvious that

$$\frac{1}{f_k}\frac{df_k}{dT} \propto \frac{dJ}{dT} . \qquad (5)$$

However, under most operating conditions the value of f_k is near unity. Thus for small changes in f_k the following relation is valid,

$$\frac{df_k}{dT} \propto \frac{dJ}{dT} \qquad (5a)$$

The left hand side of Eq. (5) is referred to as the temperature coefficient of reactivity and provides a functional dependence upon the J function and, in turn, upon the changes in temperature of the reactor.

In a paper by A. Reichel the temperature dependence of the function was investigated [2]. The results were based upon evidence gathered from experimentation with U^{238} and from analytic investigations

into the nature of the J function. Reichel concluded that physically realizable states of the reactor must be located in the range

$.02 < \xi < .6$ and $.0001 < \beta < 10$.

The results are most simply presented by Figure 1. The variable β is described by K where $K = \frac{\ln(10^5 \beta)}{\ln(2)}$. The shaded region represents states which are physically realizable for U^{238}. From the analytic investigations it was observed that values of β at the extremes of its range coincided with conditions which required that the J function be a constant. Therefore, the temperature dependence of J is destroyed and, in turn, the Doppler effect and temperature dependence of the coefficient of reactivity is nonexistant. In other regions of Figure 1 the dependence of J was found to vary with temperature raised to a constant power. These regions are indicated on Figure 1 by the lines labeled with T^{const}. Thus for any particular state of operation defined by values of β and ξ the dependence of upon temperature may be defined. If Eq. (6) is evoked, this J dependence is transferred to the governing equations, Eqs. (3) and (4), via the temperature coefficient of reactivity.

As may be noted from Figure 1 the variance of the J function with temperature may take on many different forms. However, in an analysis presented by Hummel and Okrent [3] the J function was found to vary between the functions

$J \propto \ln T$ $\qquad$ $J \propto T^{-1/2}$

$J(\xi, \beta)$ FUNCTION TABLE

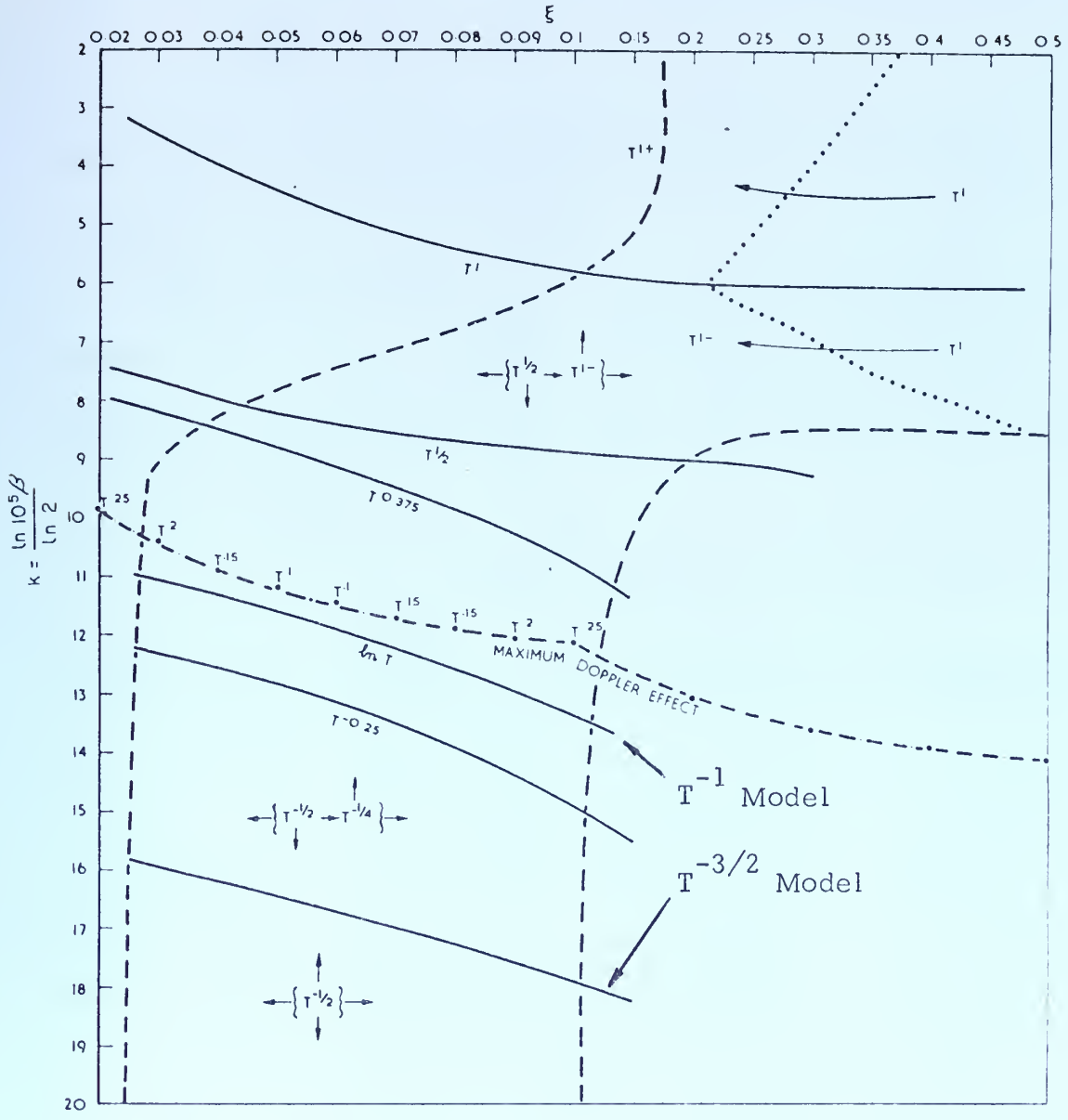

Figure 1

These results are used to determine the functional dependence of ρ_k with temperature. The results are that

$$\text{and} \qquad \frac{d\rho_k}{dT} \propto T^{-1} \qquad \frac{d\rho_k}{dT} \propto T^{-3/2} \tag{6}$$

respectively.[1] In the remainder of this paper the effect of these types of temperature feedback are analyzed.

The proportionalities in Eq. (6) are removed by the addition of a constant. The values of $\frac{d\rho_k}{dT}$ are determined by experiments performed on the reactor in question. Some representative values are given for various reactors.

TABLE 1 [4]

Reactor	Nuetron Lifetime	Temperature	Coefficient
Fermi	.14 μ sec	-3.4×10^{-5}	Δ k/k per °C
EBR II	.68 μ sec	-4.2×10^{-5}	Δ k/k per °C
Hallam	.37 m sec	$+ .3 \times 10^{-5}$	Δ k/k per °F
Peach Bottom	.23 m sec	-2.6×10^{-5}	Δ k/k per °C
EGCR	.40 m sec	-4.5×10^{-5}	Δ k/k per °C
EBOR	.22 μ sec	-1.5×10^{-5}	Δ k/k per °C

[1] In the subsequent analysis the need to differentiate between the feedback models will arise. The distinction will be made by referring to the "T^{-1} model" or "$T^{-3/2}$ model."

For the T^{-1} and the $T^{-3/2}$ model reactors the temperature dependence of the multiplication factor may be described by the equalities

$$\frac{dk}{dT} = -\frac{a'}{T}$$

and

$$\frac{dk}{dT} = -\frac{\tilde{a}}{T^{3/2}} \qquad (7)$$

respectively. It may be noted that the negative sign is incorporated into above equation. This convention is generally observed because of the greater frequency of reactors with negative temperature coefficients. In Eq. (7) the units of a' and $\tilde{a}$ are defined such that dk/dT has the units of Δ k/k per °C.

C. ASSUMPTIONS

Before the analysis of the reactor may be undertaken two further assumptions must be made concerning the dependence of the neutron flux with temperature and the delayed neutron concentration with time. The first assumption states that the reactor temperature rises instantaneously with the flux (this is the usual prompt feedback model). Equation (7) becomes

$$\frac{dk}{d\varphi} = -\frac{a}{\varphi}$$

and

$$\frac{dk}{d\varphi} = -\frac{\tilde{a}}{\varphi^{3/2}} \qquad (8)$$

The second assumption states that the delayed neutron concentration has delayed neutron concentration has reached its steady-state value.

TABLE 2 [1]

Delayed Neutron Group	Half life (sec)
Br^{87}	54.5
I^{137}	24.4
Br^{88}	16.3
I^{138}	6.3
Br^{89}	4.4
$Rb^{93,94}$	$\sim$ 6

The justification of this assumption lies in the comparison of the neutron life time in Table 1 with the precursor half-life in Table 2. It is evident that a number of neutron generations will have progressed before the precursor concentration has been changed.

The assumption of steady-state precursor concentration reduces the neutronic equation to the case without delayed neutrons. This work, therefore, deals with the case where either no delayed neutrons exist in the reactor or the delayed neutron concentration instantaneously follows the flux distribution. The latter case is realistic when considering the reactor response to small perturbations.

II. ANALYSIS OF THE REACTOR WITH A TEMPERATURE COEFFICIENT

$$\frac{dk}{dT} = -\frac{a}{T} \qquad (T^{-1} \text{ Model})$$

The solution of the nonlinear equations, Eqs. (3), (4) and (8), governing the reactor kinetics is accomplished by means of the Poincaré-Lindstedt perturbation technique [5]. This method offers a solution to the neutron flux distribution for small perturbations in the flux introduced by the temperature feedback effect. The perturbed flux distribution, referred to as $\varphi(x, \epsilon)$ is described by a Taylor series expansion about the initial state $\varphi_0(x)$ where ϵ is the amount of perturbation from $\varphi_0(x)$. The perturbation technique yields multiple equilibrium solutions for $\varphi(x, \epsilon)$ and associated eigenvalues for each of these states. The presence of more than one equilibrium state, known as the bifurcation phenomenon, requires the investigation of the stability of each of the possible states of flux. Stability considerations are made by imposing an initial deviation on an equilibrium state and examining the resulting temporal behavior of this state.

A. EQUILIBRIUM STATES FOR A SLAB REACTOR

Before the general solution to Eqs. (3), (4) and (8) is undertaken the equilibrium states for the slab reactor is investigated. For the case where the flux does not vary with time (i.e. $\frac{\partial \varphi}{\partial \tau} = 0$) and the

reactor kinetics are not dependent upon delayed neutrons the governing equations reduce to the following boundary-value problem in the spacial domain D[-1,1]:

$$\nabla^2 \varphi(x) + (k(\varphi) - 1)\varphi(x) = 0 \quad \text{in } D$$
$$\varphi(x) = 0 \quad \text{on } \partial D$$

and

$$\frac{d\,k(\varphi)}{d\,\varphi} = -\frac{a}{\varphi}$$
$$k(\varphi_0) = k_0 . \tag{8}$$

Equation (8) may be solved to give

$$k(\varphi) = k_0 - a \ln\left[\frac{\varphi(x)}{\varphi_0(x)}\right] \tag{9}$$

This result may be substituted into the boundary-value problem and the result is

$$\nabla^2 \varphi(x) + \left(\lambda - a \ln\left[\frac{\varphi(x)}{\varphi_0(x)}\right]\right)\varphi(x) = 0 \quad \text{in } D$$
$$\varphi(x) = 0 \quad \text{on } \partial D \tag{10}$$

where

$$\lambda = k_0 - 1$$

The solution to Eq. (10) is constructed about a flux distribution $\varphi_0(x)$ and eigenvalue λ_0 which will be determined. Physically, λ_0 is this eigenvalue corresponding to the initial material buckling of the reactor before perturbation by feedback. After

the system is perturbed the flux and the material buckling establish a new state which may be described by a Taylor series expansion about the initial state. If the amount of perturbation is measured by ϵ, the resultant reactor state is

$$\varphi(x,\epsilon) = \varphi_0(x) + \epsilon\,\dot{\varphi}(x) + \tfrac{1}{2}\epsilon^2\,\ddot{\varphi}(x) + \cdots$$
$$\lambda(\epsilon) = \lambda_0 + \epsilon\,\dot{\lambda} + \tfrac{1}{2}\epsilon^2\,\ddot{\lambda} + \cdots . \tag{11}$$

Thus for small perturbations of the system the values of $\varphi(x,\epsilon)$ and $\lambda(\epsilon)$ may adequately be described by truncating the series after the third term. The unperturbed flux, $\varphi_0(x)$, and eigenvalues, λ_0, are defined as the solution to the linear problem

$$\nabla^2\varphi_0(x) + (\lambda_0 - a)\,\varphi_0(x) = 0 \qquad \text{in } D$$
$$\varphi_0(x) = 0 \qquad \text{on } SD. \tag{12}$$

Since λ_0 and a are constants, the solution is

$$\varphi_0(x) = A_m \cos\sqrt{\lambda_0 - a}\;x$$
and
$$\lambda_0 - a = (m\pi/2)^2 \tag{13}$$

where $m = 1, 3, 5, \ldots$. For the purpose of simplifying the subsequent analysis the shape function ψ_m for the slab reactor is defined as

$$\psi_m = \cos(m\pi/2\;x) \quad . \tag{14}$$

The remaining terms of the expansion given by Eq. (11) are determined by differentiating Eq. (10) with respect to ϵ and taking the limit as ϵ tends to zero. The values of $\dot{\varphi}$ and $\dot{\lambda}$ are then determined. By repeating this process the values of $\ddot{\varphi}$ and $\ddot{\lambda}$ are evaluated. Thus, differentiating Eq. (10) once yields

$$\nabla^2 \dot{\varphi} + \dot{\lambda} \varphi + \lambda \dot{\varphi} - a \dot{\varphi} \ln \varphi/\varphi_0 - a \dot{\varphi} = 0. \quad (15)$$

The limit as ϵ tends to zero requires that

$$\lim_{\epsilon \to 0} \varphi(x,\epsilon) = \varphi_0(x)$$

$$\lim_{\epsilon \to 0} \lambda(\epsilon) = \lambda_0 \quad .$$

As ϵ tends to zero, Eq. (15) reduces to

$$\nabla^2 \dot{\varphi}(x) + (\lambda_0 - a)\, \dot{\varphi}(x) = -\dot{\lambda} \varphi_0(x) \quad \text{in } D \quad (16)$$

$$\dot{\varphi}(x) = 0 \quad \text{on } \delta D.$$

If Eq. (16) is to have a nontrivial solution, the solvability condition (provided in Appendix A) may be evoked. Thus

$$\int_D \left| -\dot{\lambda} \varphi_0(x) \right| \dot{\varphi}^H(x)\, dx = 0$$

where $\dot{\varphi}^H(x)$ is the homogeneous solution to Eq. (16). The solution for $\dot{\varphi}^H(x)$ is easily seen to be $\dot{\varphi}^H(x) = B_m^H \psi_m$.

When the functions $\dot{\varphi}^{H}(x)$ and $\varphi_0(x)$ are substituted into the solvability condition the result is

$$-\dot{\lambda} \int_D A_m B_m^H \psi_m^2 \, dx = 0 .$$

Since the integral is nonzero, it follows that $\dot{\lambda} = 0$. This result is substituted into Eq. (16) and the solution for $\dot{\varphi}(x)$ is clearly seen to be $\dot{\varphi}(x) = B_m \psi_m$. Thus the linear terms of the expansions are

$$\dot{\varphi}(x) = B_m \psi_m(x) \quad \text{and}$$

$$\dot{\lambda} = 0 .$$

The parameter ϵ is chosen to represent the average magnitude of the linear term in the expansion of Eq. (11), weighted by the ψ_m distribution, and measured in units of the maximum initial flux. Thus,

$$\epsilon = \frac{1}{A_m} \int_D \psi_m(x) \left(\varphi(x, \epsilon) - \varphi_0(x) \right) dx . \qquad (17)$$

Differentiating Eq. (17) with respect to ϵ yields

$$A_m = \int_D \psi_m(x) \quad \dot{\varphi}(x, \epsilon) \quad dx$$

where

$$\dot{\varphi}(x, \epsilon) = \dot{\varphi}(x) + \epsilon \, \ddot{\varphi}(x) + \cdots .$$

By letting ϵ approach zero

$$A_n = B_n \int_D \psi_n^2(x)\, dx$$

thus $A_n = B_n$. The second terms of the expansions are

$$\dot{\varphi} = A_n \psi_n \quad \text{and} \quad \dot{\lambda} = 0 . \tag{18}$$

The third term of $\varphi(x,\epsilon)$ and $\lambda(\epsilon)$ are similarly determined by differentiating Eq. (16) with respect to ϵ. The result is

$$\nabla^2 \ddot{\varphi}(x,\epsilon) + \ddot{\lambda}(\epsilon)\,\varphi(x,\epsilon) + 2\dot{\lambda}(\epsilon)\,\dot{\varphi}(x,\epsilon) + \lambda(\epsilon)\,\ddot{\varphi}(x,\epsilon) - a\,\ddot{\varphi}(x,\epsilon)\,\ln\left(\frac{\varphi(x,\epsilon)}{\varphi_0(x)}\right) - a\,\dot{\varphi}^2(x,\epsilon)\,\frac{1}{\varphi(x,\epsilon)} - a\,\ddot{\varphi}(x,\epsilon) = 0 \qquad \text{in } D$$

$$\ddot{\varphi}(x,\epsilon) = 0 \qquad \text{on } SD.$$

Since $\dot{\lambda} = 0$, the result of taking the limit as ϵ tends to zero is

$$\nabla^2 \ddot{\varphi}(x) + (\lambda_0 - a)\,\ddot{\varphi}(x) = a\,\dot{\varphi}^2(x)\,\frac{1}{\varphi_0(x)} - \ddot{\lambda}\,\varphi_0(x) \qquad \text{in } D \tag{19}$$

$$\ddot{\varphi}(x) = 0 \qquad \text{on } SD.$$

The solvability condition requires that

$$\int_D \left| a\,\dot{\varphi}^2(x)\,\frac{1}{\varphi_0(x)} - \ddot{\lambda}\,\varphi_0(x) \right| \ddot{\varphi}^H(x)\, dx = 0$$

where $\ddot{\varphi}^{H}(x)$ is the homogeneous solution to Eq. (19). This solution is obviously seen to be $\ddot{\varphi}^{H}(x) = C^{H}\,\psi_m$.

When the various functions are substituted into the solvability condition the following becomes evident

$$\ddot{\lambda}\, A_m C_m^H \int_D \psi_m^2\, dx = a\, A_m C_m^H \int_D \psi^2 dx$$

and $\ddot{\lambda} = a$. This result is replaced in Eq. (19) and it is found that the right hand side identically becomes zero. Therefore, the solution for the third terms of the expansions is $\ddot{\varphi}(x) = C_m \psi_m$ and (20) $\ddot{\lambda} = a$. The equilibrium solution for $\varphi(x,\epsilon)$ is obtained by placing the results of Eqs. (13), (18), and (20) into the series of Eq. (11). Thus,

$$\varphi(x,\epsilon) = A_m \left(1 + \epsilon + \tfrac{1}{2}\epsilon^2 \frac{C_m}{A_m} + \cdots\right) \psi_m(x) \qquad (21)$$

and

$$\lambda(\epsilon) - a = (m\pi/2)^2 + \tfrac{1}{2}\epsilon^2 a + \cdots$$

where $m = 1, 3, 5 \ldots$. The value of $\varphi(x,\epsilon)$ from Eq. (21) may be evaluated if the constants A_1 and C_1 are defined. Since the reactor is assumed to operate at its fundamental mode before temperature feedback becomes effective (later it will be shown that the other modes are unstable), the value of A_1 may be determined by the power requirements

of the reactor. Thus,

$$P = q \int_D \varphi(x)\, dx$$

where P is the linear reactor power level and q is a conversion factor. The value C_m is undetermined and will remain an unknown of order $\epsilon^2 C_m$ throughout the analysis. For each mode, m, $\varphi(x,\epsilon)$ as defined in Eq. (21), describes two possible states in the reactor as depicted in Figure 2 for the fundamental mode. The state of larger flux is determined by a positive value of ϵ while the smaller state is described by a negative value of ϵ. Associated with each state of the reactor is an eigenvalue, $\lambda(\epsilon) - a$, given by Eq. (21) and plotted against ϵ in Figure 3. The values of $\lambda(\epsilon) - a$ for the case of negative reactivity are drawn as a solid parabola which opens to the right. It may then be noted that the equilibrium state may only exist for values of $\lambda(\epsilon) - a$ which are greater than the value of λ_o, the initial material buckling of the unperturbed system. For the reactor with positive reactivity the converse is apparent. The corresponding curve for positive reactivity in Figure 3 is that drawn with a dashed line. These eigenvalues are found to be always less than the value of λ_o. It may also be noted that the eigenvalues of the system are $\lambda(\epsilon) - a$ where the term $(-a)$ describes the amount the material buckling changes so as to accommodate the temperature feedback.

FLUX DISTRIBUTION FOR SLAB REACTOR

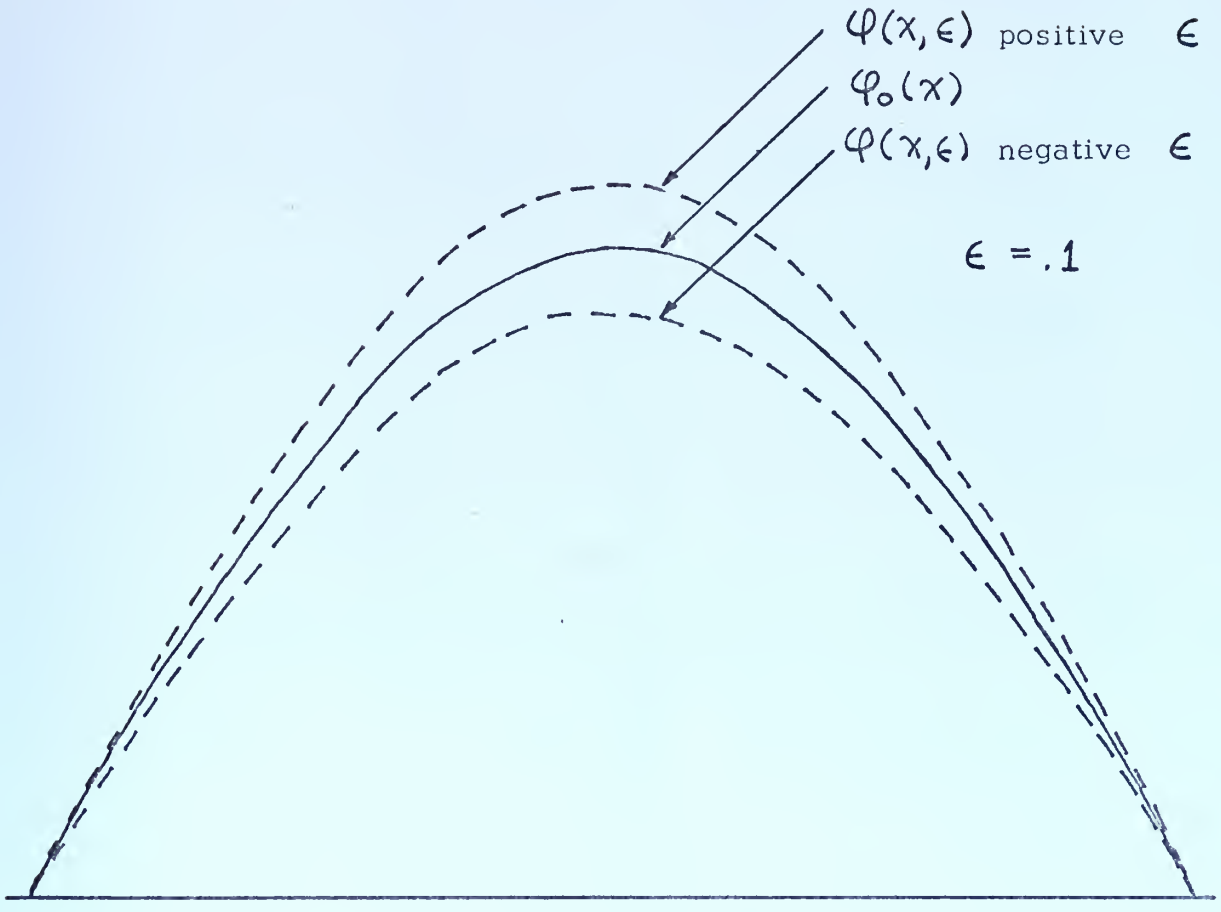

Figure 2

REACTOR BUCKLING

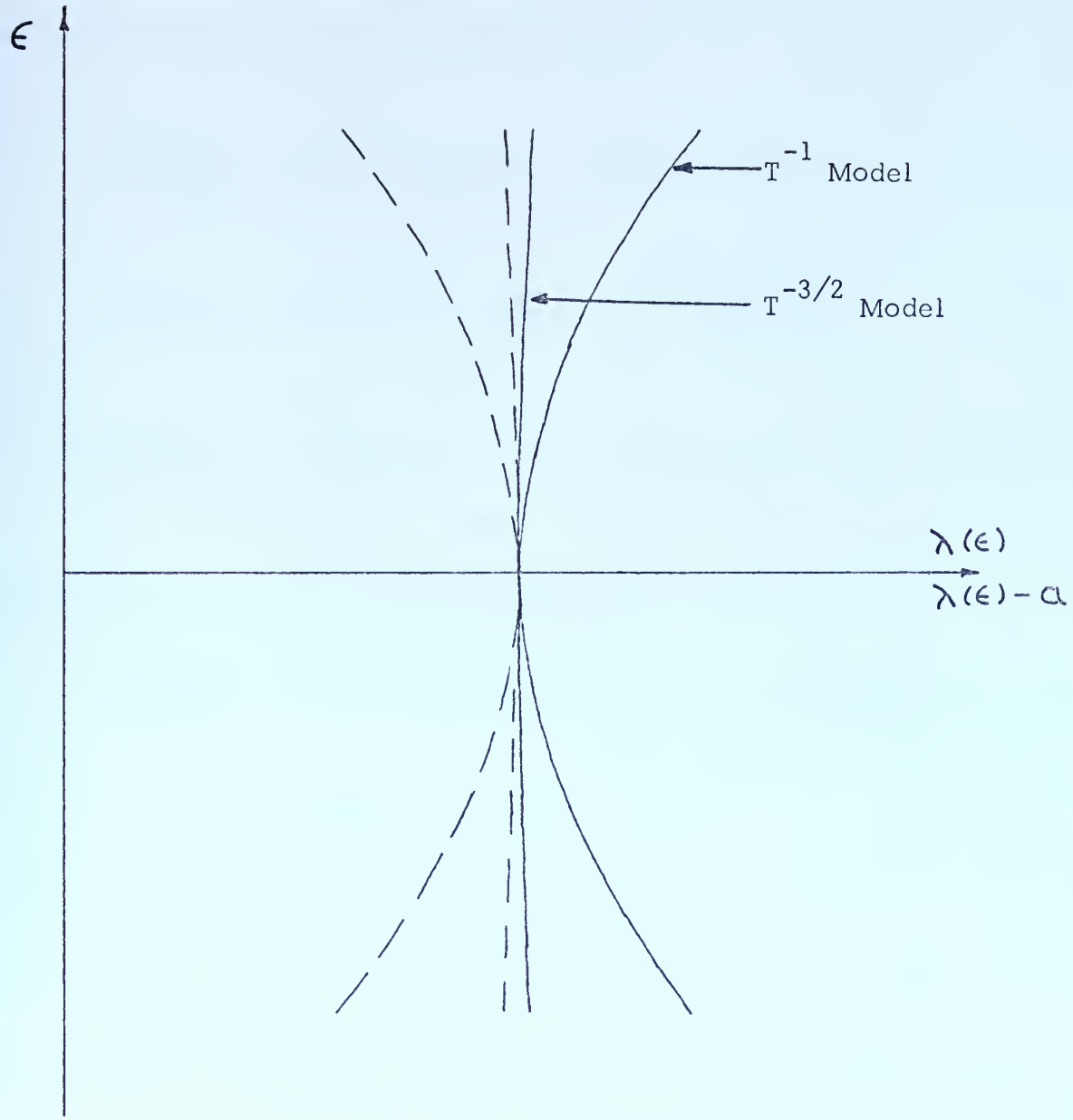

Figure 3

- - - - negative reactivity

_____ positive reactivity

B. STABILITY ANALYSIS OF THE SLAB REACTOR

The stability criterion for a slab reactor is developed by considering the temporal effect of an initial deviation on $\varphi(x,\epsilon)$. Under these conditions the governing equations, Eqs. (3), (4) and (9) reduce to

$$\frac{\partial\varphi(x,\tau)}{\partial\tau} = \nabla^2\varphi(x,\tau) + \left(k(\varphi)-1\right)\varphi(x,\tau) \qquad \text{in } D$$

$$k(\varphi) = k_0 - a\ln\left(\frac{\varphi(x,\tau)}{\varphi_0(x)}\right) \qquad \text{in } D \tag{9}$$

$$\varphi(x,0) = \varphi_0(x) \quad ; \quad \varphi(x,\tau) = 0 \qquad \text{on } \partial D.$$

These equations may be compacted into the more convenient form

$$\frac{\partial\varphi(x,\tau)}{\partial\tau} = \nabla^2\varphi(x,\tau) + \left[\lambda - a\ln\left(\frac{\varphi(x,\tau)}{\varphi_0(x)}\right)\right]\varphi(x,\tau) \qquad \text{in } D \tag{22}$$

$$\varphi(x,\tau) = 0 \qquad \text{on } \partial D$$

where

$$\lambda = k_0 - 1 .$$

The time dependent nature of the neutron flux is investigated by prescribing an arbitrary initial deviation, η , on the equilibrium flux and then observing the growth of the deviation. An unstable state of flux is that which will result in an unbounded growth of the deviation. A stable system will behave in a neutrally stable manner which implies

that the deviation tends to a new equilibrium state or in an asymptotically stable manner which implies that the deviation decays to zero. The time dependent flux is described by a Taylor series expansion about the equilibrium state $\varphi(x,\epsilon)$, defined by Eq. (21),

$$\varphi(x,\epsilon,\eta,\tau) = \varphi(x,\epsilon) + \eta\,\varphi_\eta(x,\epsilon,\tau) + \tfrac{1}{2}\eta^2\varphi_{\eta\eta}(x,\epsilon,\tau) + \cdots \tag{23}$$

such that

$$\lim_{\eta\to 0} \varphi(x,\epsilon,\tau,\eta) = \varphi(x,\epsilon) .$$

Equation (22) is differentiated with respect to ϵ. Therefore,

$$\frac{\partial \varphi_\eta}{\partial \tau} = \nabla^2\varphi_\eta + \left(\lambda(\epsilon) - a - a\ln\frac{\varphi(x,\tau)}{\varphi_0(x)}\right)\varphi_\eta \quad \text{in } D$$

$$\varphi_\eta = 0 \quad \text{on } SD$$

where

$$\varphi_\eta = \varphi_\eta(x,\epsilon,\tau,\eta) .$$

The limit as η tends to zero is taken and the result is

$$\frac{\partial \varphi_\eta(x,\epsilon,\tau)}{\partial \tau} = \nabla^2\varphi_\eta(x,\epsilon,\tau) + \left(\lambda(\epsilon) - a - a\ln\frac{\varphi(x,\tau)}{\varphi_0(x)}\right)\varphi_\eta(x,\epsilon,\tau) \quad \text{in } D \tag{24}$$

$$\varphi_\eta(x,\epsilon,\tau) = 0 \quad \text{on } SD .$$

If a solution for $\varphi_\eta(x,\epsilon,\tau)$ is chosen to be

$$\varphi_\eta(x,\epsilon,\tau) = u(x,\epsilon)\ \exp(\alpha(\epsilon)\tau) \tag{25}$$

then the stability of the system is analyzed by inquiring whether the value of $\alpha(\epsilon)$ is negative or positive. A form for $\alpha(\epsilon)$ is given by the series

$$\alpha(\epsilon) = \alpha_0 + \epsilon\dot{\alpha} + \tfrac{1}{2}\epsilon^2\ddot{\alpha} + \cdots . \tag{26}$$

The value of $\alpha(\epsilon)$ is determined by the perturbation technique previously used. In this vein, Eq. (25) is substituted into Eq. (24) and the resulting equation is

$$\nabla^2 u(x,\epsilon) + \left(\lambda(\epsilon) - a - \alpha(\epsilon) - a\ln\frac{\varphi(x,\epsilon)}{\varphi_0(x)}\right) u(x,\epsilon) = 0 \quad \text{in } D \tag{27}$$

$$u(x,\epsilon) = 0 \quad \text{on } \delta D.$$

Setting ϵ equal to zero

$$\nabla^2 u(x,\epsilon) - (\lambda_{om} - a - \alpha_0)\, u(x,0) = 0 \quad \text{in } D \tag{28}$$

$$u(x,0) = 0 \quad \text{on } \delta D.$$

Since the values of λ_{om} and (a) have been prescribed by Eq. (13),

Eq. (28) may have a nontrivial solution only if (proof is given in Appendix C)

$$\lambda_{om} - a = \lambda_{on} - a - \alpha_o \quad .$$

Thus the value of α_o is defined as

$$\alpha_{omn} = (\lambda_{on} - a) - (\lambda_{om} - a) \qquad (29)$$
$$= (\pi/2)^2 (n^2 - m^2)$$

for odd intergers of n and m where m is the mode of the deviation. If the fundamental mode of flux distribution, $n = 1$, is considered, it is evident that Eq. (29) reduces to

$$\alpha_{o1m} = (\pi/2)^2 (1 - m^2)$$

Thus the values of α_{o1m} are

$$\alpha_{o1m} = 0 \qquad m = 1$$
$$\leq -2\pi \qquad m = 3, 5, 7, \cdots .$$

For values of ϵ near zero the first term of the series for $\alpha(\epsilon)$ indicates that the fundamental mode is not unstable for any value m. However, the neutral stability predicted by $m = 1$ obliges an investigation into the higher order terms of Eq. (26). Some conclusive results may be obtained by considering the linear term of $\alpha(\epsilon)$.

For higher modes of the reactor (i.e., $m = 1$) Eq. (29) indicates that

$$\begin{aligned} \alpha(\epsilon) &\leq -2\pi^2 && m > n \\ &= 0 && m = n \\ &\geq 2\pi^2 && m < n \end{aligned}$$

Since a positive value of $\alpha(\epsilon)$ may occur, it is concluded that the higher modes of the reactor are unstable. For a physically realistic situation, however, only the fundamental mode exists in the reactor before the perturbation.

The solution for $u(x,0)$ is obtained by substituting the value of Eq. (29) into Eq. (28). The result is that

$$\nabla^2 u(x,0) + (\lambda_{0m} - a)\, u(x,0) = 0 \quad \text{in } D \tag{30}$$

$$u(x,0) = 0 \quad \text{on } \delta D.$$

It is then obvious that

$$u(x,0) = U_m \psi_m$$

where the index m designates the modes of the deviation. The value $m = 1$ refers to the fundamental mode of the deviation while $m > 1$ refers to the higher modes.

The linear term of the expansion for $\alpha(\epsilon)$, defined in Eq. (26), is derived by differentiating Eq. (27) with respect to ϵ; thus

$$\nabla^2 \dot{u}(x,\epsilon) + \left(\lambda(\epsilon) - a - \alpha(\epsilon) - a \ln \frac{\varphi(x,\epsilon)}{\varphi_o(x)}\right) \dot{u}(x,\epsilon) + \left(\dot{\lambda}(\epsilon) - \dot{\alpha}(\epsilon) - a \frac{\dot{\varphi}(x,\epsilon)}{\varphi_o(x)}\right) u(x,\epsilon) = 0 \quad \text{in } D$$

$$\dot{u}(x,\epsilon) = 0 \quad \text{on } SD. \tag{31}$$

The limit as ϵ tends to zero is taken and

$$\nabla^2 \dot{u}(x,0) + (\lambda_{om} - a - \alpha_{omm}) \dot{u}(x,0) = \left(-\dot{\lambda}(0) + \dot{\alpha}(0) + a \frac{\dot{\varphi}(x)}{\varphi_o(x)}\right) u(x,0) \quad \text{in } D$$

$$\dot{u}(x,0) = 0 \quad \text{on } SD. \tag{32}$$

By noting that $\dot{\lambda} = 0$ the solvability condition for Eq. (32) requires that

$$\int_D \left(\dot{\alpha} + a \frac{\dot{\varphi}(x)}{\varphi_o(x)}\right) u(x,0)\, \dot{u}^H(x,0)\, dx = 0$$

where $\dot{u}^H(x,0)$ is the homogeneous solution to Eq. (32). When

the value of $\alpha_{om\,m}$, from Eq. (29), is employed the function $\dot{u}^{H}(x,0)$ is determined to be

$$\dot{u}^{H}_{(x,0)} = \dot{U}^{H}_{m}\,\psi_{m} \,.$$

The value of $\dot{\alpha}$ is defined by substituting the functions $\varphi_{o}(x)$, $\dot{\varphi}(x)$ (from Eqs. (13) and (18)), $u(x,0)$, and $\dot{u}^{H}(x,0)$ into the solvability condition and the result is

$$\dot{\alpha} = -a \,. \tag{33}$$

This result and Eq. (29) determine the linear approximation to $\alpha(\epsilon)$, Eq. (26). Thus,

$$\alpha(\epsilon) = \left(\pi/2\right)^{2}\left(n^{2}-m^{2}\right) - \epsilon a$$

It now becomes evident that the question posed concerning the stability of the fundamental mode ($n=1$) for sufficiently small ϵ is answered by the linear approximation for $\alpha(\epsilon)$. Since values of $m > 1$ give very large, negative values of $\alpha(\epsilon)$ and thereby assure the stability of these modes, the concern is centered about $m = n = 1$. For this case

$$\alpha(\epsilon) = -\epsilon a$$

and the stability of the fundamental mode will depend upon the sign

of the linear term of $\alpha(\epsilon)$. The various cases to be analyzed are the following:

a) negative reactivity

$\alpha(\epsilon) < 0$ for positive ϵ

> 0 for negative ϵ

These equilibrium states of flux are depicted in Figure 2 and the corresponding eigenvalues are shown by the solid curve in Figure 3. A positive value of ϵ describes an equilibrium state of flux which is larger than the initial flux. This state, Eq. (21), exists for an eigenvalue, $\lambda(\epsilon) - a$, greater than the initial material buckling of the unperturbed reactor, λ_o , and it is observed to be stable. A negative value of ϵ indicates an unstable equilibrium state in the reactor. However, Eq. (21) indicates that the eigenvalues must be greater than the value of λ_o for any value of ϵ . Therefore, a situation in the reactor which permits a decrease in flux (i.e., negative ϵ) with an associated increase of the reactor buckling is physically unacceptable.

b) positive reactivity (evaluated by changing the sign of all terms containing a)

$\alpha(\epsilon) > 0$ for positive ϵ

< 0 for negative ϵ

These states of flux are depicted in Figure 2 and eigenvalues shown by the dashed line in Figure 3 are analyzed in a similar manner. The stable state of flux is that described by a negative value of ϵ . From Eq. (21), noting that the sign of a must be changed to accommodate positive reactivity, the fundamental eigenvalue is less than that of the initial material buckling, λ_o . Therefore, a decrease in flux (i.e., negative ϵ) is compatible with a decrease in the eigenvalues. However, for the case of positive ϵ the equilibrium state is unstable. Again it is noted that the increase of flux associated with a decrease in the material buckling is uncompatible and this situation is physically unrealistic.

The solution to Eq. (32) can be completed by first substituting Eq. (33) into Eq. (32). The right hand side of this equation becomes:

$$RHS = \left(\dot{\alpha}(0) + a\,\frac{\dot{\varphi}(x)}{\varphi_o(x)}\right) u(x,0) .$$

However, from Eqs. (13), (18), and (30); $\varphi_o(x) = A_m \psi_m$, $\dot{\varphi}(x) = A_m \psi_m$, and $u(x,0) = U_m \psi_m$ respectively. Thus,

$$RHS = (-a + a(1))\, U_m \psi_m$$
$$= 0 .$$

Equation (32) correspondingly reduces to the form

$$\nabla^2 \dot{u}(x,0) + (\lambda_{om} - a - \alpha_{omm})\dot{u}(x,0) = 0 \quad \text{in } D$$

$$\dot{u}(x,0) = 0 \quad \text{on } \delta D.$$

Thus, the solution to $\dot{u}(x,0)$ is

$$\dot{u}(x,0) = \dot{u}^H(x,0) = \dot{u}_m \psi_m . \tag{34}$$

For larger arguments of ϵ the third term of the expansion of $\alpha(\epsilon)$, Eq. (26), may become important. This term may be evaluated by differentiating Eq. (31) with respect to ϵ and then taking the limit as ϵ tends to zero. The differentiation yields

$$\nabla^2 \ddot{u}(x,\epsilon) + \left(\lambda(\epsilon) - a - \alpha(\epsilon) - a \ln \frac{\varphi(x,\epsilon)}{\varphi_0(x)}\right)\ddot{u}(x,\epsilon) + 2\left(\dot{\lambda}(\epsilon) - \dot{\alpha}(\epsilon) - a \frac{\dot{\varphi}(x,\epsilon)}{\varphi(x,\epsilon)}\right)\dot{u}(x,\epsilon) + \left(\ddot{\lambda}(\epsilon) - \ddot{\alpha}(\epsilon) - a \frac{\ddot{\varphi}(x,\epsilon)}{\varphi(x,\epsilon)} + a \frac{\dot{\varphi}^2(x,\epsilon)}{\varphi^2(x,\epsilon)}\right)u(x,\epsilon) = 0 \quad \text{in } D$$

$$\ddot{u}(x,\epsilon) = 0 \quad \text{on } \delta D .$$

When the limit as ϵ tends to zero is taken and the various known functions are substituted into the resulting equation, the following becomes evident.

$$\nabla^2 \ddot{u}(x,0) + (\lambda_{om} - a - \alpha_{omm})\,\ddot{u}(x,0) = \tag{35}$$

$$\left[\ddot{\alpha} - a\left(2 - \frac{c_m}{A_m}\right)\right] u_m \psi_m \qquad \text{in } D$$

$$\ddot{u}(x,0) = 0 \qquad \text{on } SD.$$

The solvability condition requires that

$$\int_D \left[\ddot{\alpha} - a\left(2 - \frac{c_m}{A_m}\right)\right] u_m \psi_m \ddot{u}^H(x,0)\, dx = 0$$

where $\ddot{u}^H(x,0)$ is the homogeneous solution to the above equation. The nontrivial solution for $\ddot{u}^H(x,0)$ is $\ddot{u}^H(x,0) = \ddot{U}^H_m \psi_m$.

Thus,

$$\left[\ddot{\alpha} - a\left(2 - \frac{c_m}{A_m}\right)\right] \int_D u_m \ddot{u}^H_m \psi_m^2\, dx = 0$$

and since the integral $\int_D \psi_m^2\, dx = 1$,

$$\ddot{\alpha} = a\left(2 - \frac{c_m}{A_m}\right). \tag{36}$$

The expansion of $\alpha(\epsilon)$, Eq. (26), may be written in the more complete form by the substitutions of Eqs. (29), (33), and (36). Thus

$$\alpha(\epsilon) = (\pi/2)^2 (n^2 - m^2) - \epsilon a + \tfrac{1}{2}\epsilon^2 a\left(2 - \frac{c_m}{A_m}\right) + \cdots. \tag{37}$$

For larger arguments of ϵ the stability of the system is contingent upon the negativity of $\alpha(\epsilon)$, defined by Eq. (37). For the

fundamental mode ($m = 1$) Eq. (37) reduces to

$$\alpha(\epsilon) = (\pi/2)^2 (1 - m^2) - \epsilon a + \tfrac{1}{2} \epsilon^2 a \left(2 - \frac{C_m}{A_m}\right) + \cdots . \tag{38}$$

The value of m of particular interest is that given by $m = 1$ since values of $m > 1$ provide a large, negative first term in Eq. (38). Thus, the stability criterion is

$$- \epsilon a + \tfrac{1}{2} \epsilon^2 a \left(2 - C_1/A_1\right) < 0$$

or

$$- \epsilon a \left(1 - \epsilon + \tfrac{1}{2} \epsilon \, C_1/A_1\right) < 0 \tag{39}$$

By inspection of Eq. (39) the quantity within the parenthesis may be seen to always be positive for small arguments of ϵ . Thus, the stability is governed by the coefficient ($- \epsilon a$) as has previously been considered. However, for cases of larger arguments of ϵ the inequality shown above may be changed. Thus, to the order of ϵ^2 the condition for stability is

$$\epsilon < \frac{1}{1 - \tfrac{1}{2} C_1/A_1} \, . \tag{40}$$

If this were violated, such large values of ϵ would render the reactor unstable. Thus, the undetermined peak amplitude of $\ddot{\varphi}(x)$, C_1, as compared to the peak amplitude of the unperturbed flux, A_1, is important when considering the stability regime.

C. EQUILIBRIUM STATES FOR A FINITE CYLINDRICAL REACTOR

The finite cylinder represents a realistic geometric configuration for a reactor. The physical and mathematical considerations which have been deduced for the slab reactor are directly applicable to the cylindrical reactor. In this section, the equilibrium states of flux, associated eigenvalues, and the stability of these states which have undergone feedback perturbation from some initial state will be studied for cylindrical geometry with azimuthal symmetry. The flux distribution must depend upon two spacial coordinates identified by r, the radial coordinate, and z, the longitudinal coordinate.

The equilibrium solution for the reactor is developed from the governing equations under the conditions that $\partial \varphi(r,z,\tau)/\partial\tau = 0$. The governing equations reduce to the form given by Eq. (10) which is rewritten below,

$$\nabla^2 \varphi(r,z,\tau) + \left(\lambda - a \ln \frac{\varphi(r,z,\tau)}{\varphi_0(r,z)}\right) \varphi(r,z,\tau) = 0 \quad \text{in } D \tag{41}$$

$$\varphi(r,z,\tau) = 0 \quad \text{on } SD.$$

The feedback-perturbed solution to Eq. (41) is evaluated by a Taylor series expansion about the initial state. If the amount of perturbation is ϵ, which will subsequently be defined, the resultant equilibrium state is

$$\varphi(r,z,\epsilon) = \varphi_0(r,z) + \epsilon\,\dot{\varphi}(r,z) + \tfrac{1}{2}\epsilon^2\,\ddot{\varphi}(r,z) + \cdots \quad (42)$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\,\dot{\lambda} + \tfrac{1}{2}\epsilon^2\,\ddot{\lambda} + \cdots .$$

The initial state, $\varphi_0(r,z)$ and λ_0, is defined as the solution to

$$\nabla^2\varphi_0(r,z) + (\lambda_0 - a)\,\varphi_0(r,z) = 0 \qquad \text{in } D \quad (43)$$

$$\varphi_0(r,z) = 0 \qquad \text{on } \delta D .$$

Equation (43) is solved by assuming a separable solution for $\varphi_0(r,z)$ given by

$$\varphi_0(r,z) = R(r)\,\mathcal{Z}(z) . \quad (44)$$

Substituting Eq. (44) into Eq. (43) yields the following:

$$R(r)\,\frac{d^2\mathcal{Z}}{dz^2} + \mathcal{Z}(z)\,\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dR(r)}{dr}\right)$$

$$+ (\lambda_0 - a)\,R(r)\,\mathcal{Z}(z) = 0$$

which may be separated into the equations

$$\frac{d}{dr}\left(r\,\frac{dR(r)}{dr}\right) + r\,\omega^2 R(r) = 0 \qquad \text{in } D$$

$$R(r) = 0 \qquad \text{on } \delta D ,$$

$$\frac{d^2\mathcal{Z}(z)}{dz^2} + \bar{\omega}^2\,\mathcal{Z}(z) = 0 \qquad \text{in } D$$

$$\mathcal{Z}(z) = 0 \qquad \text{on } \delta D ,$$

and $$\bar{\omega}^2 + \omega^2 = \lambda_o - a .$$

The solutions to these equations are the following:

$$R(r) = A_m^R \; J_o(\mu_m r/r_o) ,$$

$$Z(z) = A_m^z \; \cos(m\pi/2 \; z) ,$$

and

$$\lambda_o - a = (\mu_m/r_o)^2 + (m\pi/2)^2$$

where μ_m are the roots of the zeroth order Bessel function. The solution to Eq. (43) is

$$\varphi_o(r,z) = R(r) \, Z(z)$$

$$= A_m \, J_o(\mu_m r/r_o) \cos(m\pi/2 \; z)$$

and $$\lambda_o - a = (\mu_m/r_o)^2 + (m\pi/2)^2 \qquad (45)$$

where $$m = 1, 3, 5, \cdots .$$

For the purpose of simplifying the subsequent analysis the shape function ψ_m is defined for the cylinder as

$$\psi_m = J_o(\mu_m r/r_o) \cos(m\pi/2 \; z). \qquad (46)$$

Thus,

$$\varphi_{om}(r,z) = A_m \psi_m . \qquad (47)$$

The remaining terms of the expansion of $\varphi(r,z,\epsilon)$ and $\lambda(\epsilon)$, given by Eq. (42), are determined by differentiating Eq. (41) with respect to ϵ and then taking the limit as ϵ tends to zero. These steps lead to the equation

$$\nabla^2 \dot{\varphi}(r,z) + (\lambda_0 - a)\,\dot{\varphi}(r,z) = -\dot{\lambda}\,\varphi_0(r,z) \quad \text{in } D$$

$$\dot{\varphi}(r,z) = 0 \quad \text{on } \delta D. \tag{48}$$

If $\dot{\varphi}(r,z)$ is to have a nontrivial solution, the solvability condition (provided in Appendix A) may be envoked. Thus,

$$\int_D -\dot{\lambda}\,\varphi_0(r,z)\,\dot{\varphi}^{H}(r,z)\,2\pi r\,dr\,dz = 0 \tag{49}$$

where $\dot{\varphi}^{H}(r,z)$ is the homogeneous solution to Eq. (48). This solution has a similar shape function as Eq. (43). Thus,

$$\dot{\varphi}^{H}(r,z) = B^{H}\,\psi_m\,. \tag{50}$$

The fundtions $\varphi_0(r,z)$ and $\dot{\varphi}^{H}(r,z)$, Eqs. (47), and (50), respectively, are substituted into Eq. (49); hence,

$$\int_D -\dot{\lambda}\,A_m\,\psi_m\,B_m^{H}\,\psi_m\,2\pi r\,dr\,dz = 0$$

or

$$-\dot{\lambda}\,A_m\,B_m^{H}\,2\pi\int_D \psi_m^2\,r\,dr\,dz = 0\,.$$

Since the integral

$$\int_D \psi_m^2 \, r \, dr \, dz = \frac{1}{2} r_o^2 \, J_1^2(\mu_m) \neq 0 \, ,$$

the solvability condition may be satisfied only if $\dot{\lambda} = 0$. This result may be substituted into Eq. (48) and the value $\dot{\varphi}(r,z)$ is the solution to

$$\nabla^2 \dot{\varphi}(r,z) + (\lambda_o - a)\, \dot{\varphi}(r,z) = 0 \quad \text{in } D$$

$$\dot{\varphi}(r,z) = 0 \quad \text{on } \delta D \, .$$

Hence,

$$\dot{\varphi}(r,z) = \dot{\varphi}^{H}(r,z) = B_m \psi_m \, . \tag{50a}$$

The parameter ϵ is chosen to represent the average magnitude of the linear term in the expansion of Eq. (42), weighted by the function $\frac{\psi_m}{\pi r_o^2 J_1^2(\mu_m)}$ and measured in units of maximum initial flux. Thus,

$$\epsilon = \frac{1}{A_m} \int_D \frac{\psi_m}{\pi r_o^2 J_1^2(\mu_m)} \left[\varphi(r,z,\epsilon) - \varphi_o(r,z) \right] 2\pi r \, dr \, dz \tag{51}$$

Differentiating Eq. (51) with respect to ϵ and letting ϵ approach zero yields

$$A_m = B_m \int_D \frac{\psi_m^2 \, 2\pi r \, dr \, dz}{\pi r_o^2 \, J_1^2(\mu_m)}$$

$$= B_m .$$

Therefore the linear terms of the expansions of $\varphi(r, z, \epsilon)$ and $\lambda(\epsilon)$ are

$$\dot{\varphi} = A_m \psi_m$$

and

$$\dot{\lambda} = 0. \tag{53}$$

The third terms of the expansion of $\varphi(r, z, \epsilon)$ and $\lambda(\epsilon)$ are determined by differentiating Eq. (41) with respect to ϵ twice and letting ϵ approach zero

$$\nabla^2 \ddot{\varphi}(r,z) + (\lambda_o - a)\, \ddot{\varphi}(r,z) = a \frac{\dot{\varphi}^2(r,z)}{\varphi_o(r,z)} - \ddot{\lambda}\, \varphi_o(r,z) \quad \text{in } D \tag{54}$$

$$\ddot{\varphi}(r,z) = 0 \quad \text{on } \partial D .$$

The solvability condition requires that

$$\int_D \left(a \frac{\dot{\varphi}^2(r,z)}{\varphi_o(r,z)} - \ddot{\lambda}\, \varphi_o(r,z) \right) \ddot{\varphi}^H(r,z) \, 2\pi r \, dr \, dz = 0$$

where $\ddot{\varphi}^H(r,z)$ is the homogeneous solution to Eq. (54). The solution for $\ddot{\varphi}^H(r,z)$ is similar to that obtained from Eq. (43), thus

$$\ddot{\varphi}^{H}(r,z) = C_m^{H} \psi_m .$$

The functions $\varphi_o(r,z)$ and $\dot{\varphi}(r,z)$, determined from Eqs. (47) and (53) respectively, and $\ddot{\varphi}^{H}(r,z)$ are substituted into the solvability condition. Thus

$$\int_D (a A_m \psi_m - \ddot{\lambda} A_m \psi_m) C_m^{H} \psi_m \, 2\pi r \, dr \, dz = 0$$

or $$(a - \ddot{\lambda}) \int_D \psi_m^2 \, r \, dr \, dz = 0 .$$

Therefore,

$$\ddot{\lambda} = a . \tag{55a}$$

If this result is placed into Eq. (54), the right hand side will identically be equal to zero. Thus the solution for $\ddot{\varphi}(r,z)$ is similar to that obtained for Eq. (43) and

$$\ddot{\varphi}(r,z) = C_m \psi_m . \tag{55b}$$

The various results for the terms of the expansion of Eq. (42) are presented as follows:

$$\varphi(r,z,\epsilon) = A_m \left(1 + \epsilon + \tfrac{1}{2}\epsilon^2 \frac{C_m}{A_m} + \cdots\right) \psi_m$$

and

$$\lambda(\epsilon) - a = \left(\mu_m / r_o\right)^2 + \left(n\pi/2\right)^2 + \tfrac{1}{2}\epsilon^2 a + \cdots . \tag{56}$$

It is apparent that for any mode of flux there will be two possible equilibrium states defined by the negative and positive values of ϵ. For each of these states a unique eigenvalue is present. For negative reactivity ($\lambda(\epsilon) - a$) must be greater than λ_0 and for positive reactivity the opposite must be apparent. It may be noted that the eigenvalues are defined by $\lambda(\epsilon) - a$. Where the term $-a$ denotes the amount of change of the material buckling to accommodate the temperature feedback.

D. STABILITY ANALYSIS OF THE FINITE CYLINDRICAL REACTOR

The stability criterion for a cylindrical reactor is developed by considering the temporal effect of a deviation on the equilibrium states. The governing equation has the form

$$\frac{\partial \varphi(r,z,\tau)}{\partial \tau} = \nabla^2 \varphi(r,z,\tau) + \left(\lambda - a \ln \frac{\varphi(r,z,\tau)}{\varphi_0(r,z)}\right) \varphi(r,z,\tau) \qquad (57)$$

in D

$$\varphi(r,z,\tau) = 0 \qquad \text{on} \quad \delta D$$

$$\varphi(r,z,0) = \varphi_0(r,z) .$$

The temporal nature of the flux is investigated by imposing an initial deviation, η, on the equilibrium states defined by Eq. (56) and then observing the growth of the deviation. An unbounded increase will indicate instability; neutral stability implies that the deviation tends to a new equilibrium state; and asymptotic stability is assured

by the decay of the deviation to zero. The deviation is described by a Taylor series expansion about the equilibrium state $\varphi(r,z,\epsilon)$, defined by Eq. (56):

$$\varphi(r,z,\epsilon,\eta,\tau) = \varphi(r,z,\epsilon) + \eta\,\varphi_\eta(r,z,\epsilon,\tau) + \tfrac{1}{2}\eta^2\varphi_{\eta\eta}(r,z,\epsilon,\tau) + \cdots \tag{58}$$

such that

$$\lim_{\eta\to 0} \varphi(r,z,\epsilon,\eta,\tau) = \varphi(r,z,\epsilon) .$$

Equation (57) is differentiated with respect to η and the limit as η tends to zero is taken. The result of this process is

$$\frac{\partial \varphi_\eta}{\partial \tau} = \nabla^2\varphi_\eta + \left(\lambda(\epsilon) - a - a\ln\frac{\varphi(r,z,\epsilon)}{\varphi_0(r,z)}\right)\varphi_\eta(r,z,\epsilon) \quad \text{in } D \tag{58a}$$

$$\varphi_\eta = \varphi_\eta(r,z,\epsilon) \qquad \varphi_\eta = 0 \qquad \text{on } \delta D .$$

If a solution for $\varphi_\eta(r,z,\epsilon,\tau)$ is chosen to be

$$\varphi_\eta(r,z,\epsilon,\tau) = U(r,z,\epsilon)\,\exp\left(\alpha(\epsilon)\,\tau\right) , \tag{58b}$$

then the stability of the system is determined by the sign of $\alpha(\epsilon)$.

A form for $\alpha(\epsilon)$ is given by the series expansion

$$\alpha(\epsilon) = \alpha_0 + \epsilon\,\dot{\alpha} + \tfrac{1}{2}\epsilon^2\,\ddot{\alpha} + \cdots . \tag{59}$$

The value of $\alpha(\epsilon)$ is determined by the perturbation technique previously used. The function given by Eq. (58b) is substituted into

Eq. (58a) and the resulting relation is

$$\nabla^2 u + \left(\lambda(\epsilon) - a - \alpha(\epsilon) - a \ln \frac{\varphi(r,z,\epsilon)}{\varphi_o(r,z)}\right) u = 0 \quad \text{in } D \tag{60}$$

$$u(r,z,\epsilon) = 0 \qquad \text{on } \delta D .$$

The value of α_o may be determined by taking the limit of Eq. (60) as ϵ tends to zero. Thus,

$$\nabla^2 u(r,z) + (\lambda_o - a - \alpha_o)\, u(r,z) = 0 \quad \text{in } D \tag{61}$$

$$u(r,z) = 0 \qquad \text{on } \delta D.$$

A nontrivial solution to this equation is obtained only if (see Appendix C for proof of this)

$$\lambda_{om} - a = \lambda_{om} - a - \alpha_o .$$

Thus α_o is given by

$$\alpha_{omm} = (\lambda_{om} - a) - (\lambda_{om} - a) \tag{62}$$
$$= \frac{1}{r_o^2}\left(\mu_m^2 - \mu_m^2\right) + \left(\pi/2\right)^2 (m^2 - m^2).$$

Some remarks may be directed toward the stability of the system via Eq. (62). As was apparent for the slab reactor the fundamental mode ($m = 1$) may be evaluated as follows:

$$\alpha_{o1m} = 0 \qquad m = 1 \tag{62a}$$
$$< -2\pi^2 \qquad m > 1 .$$

Thus the higher modes of deviation ($m > 1$) which are imposed upon the equilibrium flux will vanish with time. However, the value of $m = 1$ provides $\alpha_{oim} = 0$ and the system may be thought of as neutrally stable. Thus it is necessary to inquire about the effect of $\dot{\alpha}$ upon the system for the value of $m = 1$.

The higher modes of the reactor ($m > 1$) are analyzed as follows:

$$\begin{aligned} \alpha(\epsilon) &< -2J^2 && m > n \\ &= 0 && m = n \\ &> 2J^2 && m < n. \end{aligned} \tag{62b}$$

Since the condition provided by $m < n$ may be realized for modes of $m > 1$, values of $\alpha(\epsilon) > 0$ are present and the higher modes are unstable. However, the reactor is assumed to be operating at the fundamental mode before the feedback is introduced. Equation (62) for α_{omm} may be used in Eq. (61) to provide a result similar to Eq. (43). Thus,

$$u(r,z) = U_m \psi_m. \tag{63}$$

(See Appendix C for the proof of this.)

The linear term of the expansion of $\alpha(\epsilon)$ (Eq. (59)) is evaluated by differentiating Eq. (60) with respect to ϵ and taking the limit as

ϵ approaches zero. The result is

$$\nabla^2 \dot{u}(r,z) + (\lambda_{om} - a - \alpha_{om\,m})\,\dot{u}(r,z) \qquad (64)$$
$$= \left(-\dot{\lambda} + \dot{\alpha} + a\,\frac{\dot{\varphi}(r,z)}{\varphi_o(r,z)}\right) u(r,z) \quad \text{in } D$$
$$\dot{u}(r,z) = 0 \quad \text{on } \delta D .$$

By noting that $\dot{\lambda}$ has previously been determined to be zero the solvability condition requires that

$$\int_D \left(\dot{\alpha} + a\,\frac{\dot{\varphi}(r,z)}{\varphi_o(r,z)}\right) u(r,z)\,\dot{u}^H(r,z) = 0 \qquad (64b)$$

where $\dot{u}^H(r,z)$ is the homogeneous solution to Eq. (64). By the same reasoning used for the solution of $u(r,z)$ the solution for $\dot{u}^H(r,z)$ is

$$\dot{u}^H(r,z) = \dot{u}_m^H \psi_m .$$

When this result and those provided by Eqs. (47), (53), and (63) are used to evaluate Eq. (64b) a value for $\dot{\alpha}$ is derived. Thus,

$$\int_D \left(\dot{\alpha} + a\,\frac{A_m \psi_m}{A_m \psi_m}\right) u_m \dot{u}_m^H \psi_m^2\, 2\pi r\, dr\, dz = 0$$

or

$$(\dot{\alpha} + a)\int_D \psi_m^2\, r\, dr\, dz = 0 .$$

Therefore,

$$\dot{\alpha} = -a. \tag{65}$$

This result may be substituted into the expansion for $\alpha(\epsilon)$ and further analysis of the stability is made. Thus,

$$\alpha(\epsilon) = \alpha_{omm} - \epsilon a . \tag{66}$$

The question pertaining to the stability of the fundamental mode $(m = 1)$ is resolved for small ϵ. For the case where $m = 1$ the value of α_{011} is zero and

$$\alpha(\epsilon) = -\epsilon a .$$

The analysis of the various cases of negative and positive feedback is similar to that developed for the slab reactor. The conditions of stability for the fundamental mode are briefly stated as follows:

a) The stable equilibrium state for a reactor with negative reactivity has a positive value of ϵ. The other equilibrium state is unrealistic.

b) The stable equilibrium state for a reactor with positive reactivity has a negative value of ϵ. The other equilibrium state is unrealistic.

If the value of $\dot{\alpha}$ is substituted into Eq. (64), the right hand side will become zero. Therefore the solution for $\dot{u}(r,z)$ is the homogeneous solution to Eq. (64) and

$$\dot{u}(r,z) = \dot{u}_m \psi_m . \tag{66a}$$

For larger values of ϵ the quadratic term of $\alpha(\epsilon)$ may be necessary. To obtain the desired results Eq. (60) is differentiated twice with respect to ϵ and the limit as ϵ tends to zero is evaluated. If all of the known functions are employed the result is

$$\nabla^2 \ddot{u}(r,z) + (\lambda_{om} - a - \alpha_{omm})\ddot{u}(r,z) = (67)$$
$$(\ddot{\alpha} - a(2 - C_m/A_m))\, U_m \psi_m \quad \text{in } D$$
$$\ddot{u}(r,z) = 0 \quad \text{on } \delta D .$$

The solvability condition requires that

$$\int [\ddot{\alpha} - a(2 - \tfrac{C_m}{A_m})]\, U_m \psi_m \ddot{u}^H(r,z)\, 2\pi r\, dr\, dz = 0$$

where $\ddot{u}^H(r,z)$ is the homogeneous solution to Eq. (67). Therefore $\ddot{u}^H = \ddot{U}^H_m \psi_m$. It is obvious that the solvability criterion is met only if

$$\ddot{\alpha} - a(2 - \tfrac{C_m}{A_m}) = 0$$

or

$$\ddot{\alpha} = a(2 - C_m/A_m) . \qquad (68)$$

This result is substituted into the expansion for $\alpha(\epsilon)$, Eq. (59), and the following becomes apparent,

$$\alpha(\epsilon) = \left(\frac{1}{r_o}\right)^2 \left(\mu_m^2 - \mu_m^2\right) + \left(\pi/2\right)^2 \left(m^2 - m^2\right) \tag{69}$$
$$- \epsilon a + \frac{1}{2} \epsilon^2 a \left(2 - {}^{C_m}/_{A_m}\right) + \cdots .$$

Since the fundamental mode of operation is generally of greatest interest, Eq. (69) is rewritten for $m = 1$.

$$\alpha(\epsilon) = \left(\frac{1}{r_o}\right)^2 \left(\mu_1^2 - \mu_m^2\right) + \left(\pi/2\right)^2 \left(1 - m^2\right) \tag{70}$$
$$- \epsilon a \left[1 - \frac{1}{2} \epsilon \left(2 - {}^{C_1}/_{A_1}\right)\right] + \cdots .$$

For $m > 1$ the first two terms dominate and $\alpha(\epsilon) < 0$. For $m = 1$ the stability may be determined by evaluating

$$\alpha(\epsilon) = - \epsilon a \left[1 - \frac{1}{2} \epsilon \left(2 - {}^{C_m}/_{A_m}\right)\right] .$$

These results are similar to those obtained for the slab reactor and stability is granted under the conditions established for Eq. (37).

III. ANALYSIS OF THE REACTOR WITH A TEMPERATURE COEFFICIENT

$$\frac{dk}{dT} = -a\,T^{-3/2} \quad (T^{-3/2} \text{ MODEL})$$

The solution of the nonlinear equations (3), (4), and (8) governing the reactor kinetics is obtained by the same perturbation technique used to analyze the "T^{-1} model." This method investigates the effect of a small perturbation introduced by the feedback effect. The equilibrium state that ensues from the perturbation is defined by a series expansion about the initial unperturbed state of the reactor. The perturbation technique offers multiple solutions and unique eigenvalues for each of the equilibrium states (see Figure 2). The presence of more than one equilibrium state, known as the bifurcation phenomenon, requires an investigation of the stability of each of these solutions. The procedure of analyzing the stability entails imposing an initial deviation on the equilibrium flux and then examining the resulting temporal behavior of the deviation.

A. EQUILIBRIUM STATES FOR THE SLAB REACTOR

The first consideration to be made in the analysis of the "$T^{-3/2}$ model" is that of determining the possible equilibrium states of the reactor after a feedback perturbation has been imposed. The solution to the governing Eqs. (3), (4), and (8) is desired for the situation where the perturbed flux has achieved the equilibrium state (i.e., $\frac{\partial \varphi(x,\tau)}{\partial \tau} = 0$).

The governing equations reduce to the form

$$\nabla^2 \varphi(x) + (k(\varphi) - 1)\,\varphi(x) = 0 \qquad \text{in } D \tag{3}$$

$$\varphi(x) = 0 \qquad \text{on } \delta D$$

and

$$\frac{dk}{d\varphi} = -\frac{\tilde{a}}{\varphi^{3/2}} \quad , \quad k(\varphi_0) = k_0 \, . \tag{8}$$

The equation for the multiplication factor is easily shown to be

$$k(\varphi) = k_0 + a\left[\varphi^{-1/2}(x) - \varphi_0^{-1/2}(x)\right] .$$

The function $\varphi_0(x)$ and the constant k_0 are considered the initial flux distribution and multiplication factor before the feedback effect becomes significant. The result is substituted into Eq. (3), yielding thereby:

$$\nabla^2 \varphi(x) + \left[\lambda + a\left(\varphi^{-1/2}(x) - \varphi_0^{-1/2}(x)\right)\right]\varphi(x) = 0 \qquad \text{in } D \tag{71}$$

$$\varphi(x) = 0 \qquad \text{on } \delta D$$

where

$$\lambda = k_0 - 1 .$$

After the perturbation due to feedback the new equilibrium state is described by a Taylor series expansion about the initial flux distribution, $\varphi_0(x)$, and the initial eigenvalues, λ_0. If the amount of perturbation is measured by ϵ (the perturbation parameter which

will subsequently be defined), the resultant reactor state is

$$\varphi(x,\epsilon) = \varphi_0(x) + \epsilon\,\dot{\varphi}(x) + \tfrac{1}{2}\epsilon^2\,\ddot{\varphi}(x) + \cdots \tag{11}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\,\dot{\lambda} + \tfrac{1}{2}\epsilon^2\,\ddot{\lambda} + \cdots .$$

For small perturbations of the system the values of $\varphi(x,\epsilon)$ and $\lambda(\epsilon)$ are described by truncating the series after the third term. The unperturbed states for the reactor is defined as the solution to the problem

$$\nabla^2 \varphi_0(x) + \lambda_0 \varphi_0(x) = 0 \qquad \text{in } D \tag{72}$$

$$\varphi_0(x) = 0 \qquad \text{on } \partial D .$$

It is readily seen that

$$\varphi_0(x) = A_m \cos\sqrt{\lambda_0}\,x$$

$$\lambda_0 = (m\pi/2)^2 \tag{72b}$$

$$m = 1, 3, 5, \ldots .$$

So as to simplify the subsequent analysis the shape function ψ_m is defined for the slab reactor as

$$\psi_m = \cos(m\pi/2)\,x . \tag{73}$$

Thus,

$$\varphi_0(x) = A_m \psi_m . \tag{74}$$

The next term of the expansion is determined by differentiating Eq. (71) with respect to ϵ and taking the limit as ϵ tends to zero. This differentiation yields

$$\nabla^2 \dot{\varphi}(x,\epsilon) + \left[\lambda(\epsilon) + a\left(\varphi^{-1/2}(x,\epsilon) - \varphi_0(x,\epsilon)\right)\right]\dot{\varphi}(x,\epsilon)$$
$$= -\left(\dot{\lambda}(\epsilon) - \frac{a}{2}\,\dot{\varphi}(x,\epsilon)\,\varphi^{-3/2}(x,\epsilon)\right)\varphi(x,\epsilon) \quad \text{in } D \tag{75}$$

$$\dot{\varphi}(x,\epsilon) = 0 \qquad \text{on } \delta D.$$

As ϵ approaches zero, $\varphi(x,\epsilon)$ approaches $\varphi_0(x)$, $\lambda(\epsilon)$ approaches λ_0, and

$$\nabla^2\dot{\varphi}(x) + \left(\lambda_0 - \frac{a}{2}\,\varphi_0^{-1/2}(x)\right)\dot{\varphi}(x) = -\dot{\lambda}\,\varphi_0(x) \quad \text{in } D \tag{76}$$
$$\dot{\varphi}(x) = 0 \qquad \text{on } \delta D.$$

In order to obtain a solution to Eq. (76) the solvability condition (given in Appendix A) requires that the

$$\int_D \left(-\dot{\lambda}\varphi_0(x)\right)\dot{\varphi}^H(x)\,dx = 0 \tag{77}$$

where $\dot{\varphi}^H(x)$ is the homogeneous solution to Eq. (76) given below

$$\nabla^2\dot{\varphi}^H(x) + \left[\lambda_0 - \frac{a}{2}\,\varphi_0^{-1/2}(x)\right]\dot{\varphi}^H(x) = 0 \quad \text{in } D \tag{78}$$
$$\dot{\varphi}^H(x) = 0 \qquad \text{on } \delta D.$$

A solution to Eq. (78) is offered by a modification to the R. Courant and D. Hilbert perturbation method (provided in Appendix B). Their approximated solution is given by

$$\dot{\varphi}^H(x) = u_m(x) + \frac{a}{2}\,v_m(x) + \left(\frac{a}{2}\right)^2 w_m(x) + \cdots. \tag{79}$$

Since the contribution of $(a/2)^2$ is very small only the first two terms of the series will be retained. The substitution of Eq. (79) into Eq. (78) leads to the requirement that $u_m(x)$ be the solution to

$$\nabla^2 u_m(x) + \lambda_{om} u_m(x) = 0 \qquad \text{in } D$$

$$u_m(x) = 0 \qquad \text{on } \delta D$$

the solution to $v_m(x)$ can be obtained as

$$v_m(x) = \sum_{\substack{j=1,3,5,\cdots \\ j \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} u_j$$

where $d_{mj} = \int_D \varphi_o^{-1/2}(x)\, u_m u_j \, dx$.

The orthonormal solution for $u_m(x)$ is $u_m(x) = \psi_m$ and $\lambda_{om} = (m\pi/2)^2$. The constant d_{mj} is determined as

$$d_{mj} = \frac{1}{\sqrt{A_m}} \int_D \psi_m^{1/2} \psi_j \, dx \; ;$$

thus

$$v_m(x) = \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \left(\frac{2}{\pi}\right)^2 \frac{d_{mj}}{m^2 - j^2} \psi_j \; .$$

These results may be combined into the expansion given by Eq. (77) to yield the following result,

$$\dot{\varphi}^H(x) = B_m^H \left[\psi_m + \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{m^2 - j^2} \psi_j + \cdots \right]. \qquad (80)$$

For the convenience of the subsequent analysis this function will be referred to as the Υ function given by

$$\Upsilon_m = \psi_m + \frac{2a}{J^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{m^2 - j^2} \psi_j + \cdots \quad (81)$$

A question may arise concerning the convergence of the series used in Eq. (81). If the Weierstrass M test for uniform convergence is employed, a series which is a continuous function of x in the domain D is convergent under the condition

$$|f_j(x)| \leq M_j$$

where $\sum_j M_j$ is a known convergent series of positive constants [6]. If the convergence of

$$f_j(x) = \frac{d_{mj}}{m^2 - j^2} \psi_j(x)$$

is doubted this test may be used. First, the maximum value of

is determined as follows:

$$\begin{aligned} d_{mj} &= \frac{1}{\sqrt{A_m}} \int_{-1}^{+1} \psi_m^{1/2} \psi_j \, dx \\ &\leq \frac{1}{\sqrt{A_m}} \int_{-1}^{+1} \psi_{m\,MAX}^{-1/2} \psi_{j\,MAX} \, dx \\ &\leq \frac{1}{\sqrt{A_m}} \int_{-1}^{+1} dx \\ &\leq 2/\sqrt{A_m} \end{aligned}$$

Since the function $\psi_j \leq 1$ throughout the domain,

$$\frac{2}{\sqrt{A_m}\,(j^2-m^2)} \geq \left| \frac{d_{mj}}{m^2-j^2}\, \psi_j(x) \right| .$$

If the convergence of the left hand side of the inequality is provided, then $f_j(x)$ is convergent by virtue of the Weierstrass M test. The integral test provides a convenient method to establish this convergence. Hence, if the integral

$$\int_J^\infty \frac{2}{\sqrt{A_m}\,(j^2-m^2)}\, dj$$

exists, its convergence and that of $\frac{d_{mj}}{m^2-j^2}\, \psi_j(x)$ is determined. Thus,

$$\int_J^\infty \frac{2}{\sqrt{A_m}\,(j^2-m^2)}\, dj = \frac{2}{\sqrt{A_m}} \left(\frac{-1}{m}\right) \coth(j/m) \Big|_J^\infty$$

$$= \frac{2}{\sqrt{A_m}} \left(\frac{-1}{m}\right) \left[1 - \coth(J/m)\right]$$

where J is any possible value of j. Since the integral exists, the series converges.

The values of $\varphi_0(x)$ and $\dot{\varphi}^H(x)$ from Eqs. (74) and (80), respectively, are substituted into Eq. (77) and $\dot{\lambda}$ is evaluated as shown.

$$\int_D -\dot{\lambda} A_m \psi_m B_m^H \left[\psi_m + \frac{2a}{J^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{m^2 - j^2} \psi_j + \cdots \right] dx = 0.$$

Because of the orthogonality of the ψ_m function the above equation reduces to the following form,

$$\int_D -\dot{\lambda} A_m B_m^H \psi_m^2 \, dx = 0.$$

This equation may be satisfied by the condition

$$\dot{\lambda} = 0. \tag{82}$$

The value of $\dot{\lambda}$ is substituted into Eq. (76) and the homogeneous equation that results is similar to that defined by $\dot{\varphi}^H(x)$. Thus,

$$\dot{\varphi}(x) = B_m \Upsilon_m. \tag{83}$$

The value of the perturbation parameter, ϵ, is chosen to represent the average magnitude of the linear term in the expansion for Eq. (11) weighted by the Υ_m function and measured in units of the maximum initial flux. Thus

$$\epsilon = \frac{1}{A_m} \int_D \Upsilon_m(x) \left[\varphi(x,\epsilon) - \varphi_0(x) \right] dx \tag{84}$$

If Eq. (84) is differentiated with respect to ϵ, the result is

$$1 = \frac{1}{A_m} \int_D \Upsilon_m(x) \left[\dot{\varphi}(x) + \epsilon \ddot{\varphi}(x) + \cdots \right] dx.$$

By letting ϵ approach zero and employing the solution for $\dot{\varphi}(x)$ (i.e., $\dot{\varphi}(x) = B_n Y$) the above equation reduces to

$$\frac{A_n}{B_n} = \int_D Y_n^2 \, dx$$

or, from equation (81)

$$\frac{A_n}{B_n} = \int_D \left[\psi_n + \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{n^2 - j^2} \psi_j + \cdots \right]^2 dx.$$

The quantity within the parenthesis is expanded and

$$\frac{A_n}{B_n} = \int_D \psi_n^2 \, dx + 2 \int_D \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{n^2 - j^2} \psi_n \psi_j \, dx$$

$$+ a^2 \int \left(\frac{2}{\pi^2}\right)^2 \left[\sum_{\substack{j=1,3,5,\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{n^2 - j^2} \psi_j + \cdots \right]^2 dx + O(a^3).$$

The second term on the right hand side is identically zero because of the orthogonality of the ψ_n function. When the integration of the first and third terms is evaluated, i.e., $\int_D \psi_n^2 dx = 1$, the result is

$$\frac{A_n}{B_n} = 1 + a^2 \left[\frac{2}{\pi^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{n^2 - j^2} \right]^2 + O(a^3).$$

Since the order of a^2 is generally very small, the contribution of

the higher order terms are neglected. Therefore,

$$A_m \simeq B_m .$$

Equation (83) that defines $\dot{\varphi}(x)$ may be rewritten as

$$\dot{\varphi}(x) = A_m \, T_m \tag{85}$$

The third terms in the expansions for $\varphi(x,\epsilon)$ and $\lambda(\epsilon)$ given in Eq. (11), are determined by differentiating Eq. (75) with respect to ϵ and then taking the limit as ϵ approaches zero. The differentiation yields

$$\nabla^2 \ddot{\varphi}(x,\epsilon) + \left[\lambda(\epsilon) + a(\varphi(x,\epsilon) - \varphi_0^{-1/2}(x))\right] \ddot{\varphi}(x,\epsilon)$$
$$+\left(\dot{\lambda}(\epsilon) - \frac{a}{2}\,\varphi^{-3/2}(x,\epsilon)\,\dot{\varphi}(x,\epsilon)\right)\dot{\varphi}(x,\epsilon) =$$
$$-\left(\dot{\lambda}(\epsilon) - \frac{a}{2}\,\dot{\varphi}(x,\epsilon)\,\varphi^{-3/2}(x,\epsilon)\right)\dot{\varphi}(x,\epsilon)$$
$$-\left(\ddot{\lambda}(\epsilon) - \frac{a}{2}\,\ddot{\varphi}(x,\epsilon)\,\varphi^{-3/2}(x,\epsilon) + \frac{3a}{4}\,\dot{\varphi}(x,\epsilon)\,\varphi^{-5/2}(x,\epsilon)\right)$$
$$\times \varphi(x,\epsilon) .$$

As ϵ approaches zero and Eq. (82) for $\dot{\lambda}(0)$ is used the above equation simplifies to the form

$$\nabla^2 \ddot{\varphi}(x) + \left(\lambda_0 - \frac{a}{2}\varphi_0^{-1/2}(x)\right)\ddot{\varphi}(x) = \tag{86}$$

$$-\ddot{\lambda}\,\varphi_0(x) + \frac{a}{4}\varphi_0^{-3/2}(x)\,\dot{\varphi}^2(x) \quad \text{in } D$$

$$\ddot{\varphi}(x) = 0 \qquad \text{on } \delta D .$$

The solvability condition for Eq. (86) is

$$\int_D \left(-\ddot{\lambda}\,\varphi_0(x) + \frac{a}{4}\varphi_0^{-3/2}(x)\,\dot{\varphi}^2(x)\right)\ddot{\varphi}^H(x)\,dx = 0 \tag{87}$$

where $\ddot{\varphi}^H(x)$, the homogeneous solution to Eq. (86), may be seen to have precisely the same shape function as the solution to Eq. (78). Thus,

$$\ddot{\varphi}^H(x) = C_m^H\,\Upsilon . \tag{88}$$

If the values of $\varphi_0(x)$, $\dot{\varphi}(x)$, and $\ddot{\varphi}^H(x)$ from Eqs. (74), (85), and (88), respectively, are substituted into the solvability condition of Eq. (87), the following becomes evident,

$$\ddot{\lambda}\,A_m C_m^H \int_D \psi_m \Upsilon_m\,dx = \frac{a}{4}A_m^{1/2} C_m^H \int_D \Upsilon_m^3/\psi_m^{3/2}\,dx .$$

The integrals are simplified using the definition of Υ_m given in Eq. (81). Thus

$$\int_D \psi_m \Gamma_m \, dx \cong \int_D \left[\psi_m^2 + \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5\cdots \\ \neq m}}^{\infty} \frac{d_m j}{m^2 - j^2} \psi_j \psi_m \right] dx$$

$$\cong \int_D \psi_m^2 \, dx = 1 .$$

because of the orthogonality of the ψ_m function and

$$\int_D \left[\Gamma_m / \psi_m^{1/2} \right]^3 dx = \int_D \left[\psi_m^{1/2} + \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_m j}{m^2 - j^2} \frac{\psi_j}{\psi_m^{1/2}} \right]^3 dx .$$

If the contribution of any term having a coefficient of a^2 or larger is considered negligible, an approximate value of $\ddot{\lambda}$ may be evaluated. Hence,

$$\ddot{\lambda} A_m C_m^H \cong \frac{a}{4} A_m^{1/2} C_m^H \int_D \psi_m^{3/2} \, dx . \qquad (89)$$

The definite integral may be evaluated for the fundamental mode of reactor operation (i.e., $m = 1$) and the value of $\ddot{\lambda}$ for $m = 1$ is

$$\ddot{\lambda} \cong \frac{a}{\sqrt{A_1}} (.279) . \qquad (90)$$

The equilibrium solution for $\varphi(x, \epsilon)$ and $\lambda(\epsilon)$ is obtained from the results of Eqs. (72), (74), (82), (85), and (90). The expansion for $\varphi(x, \epsilon)$ and $\lambda(\epsilon)$, equation (11) is

$$\varphi(x, \epsilon) = A_m \left(\psi_m + \epsilon \Gamma_m + O(\epsilon^2) \right) . \qquad (91)$$

where ψ_m and Γ_m are defined by Eqs. (73) and (81), respectively, and for the fundamental mode

$$\lambda(\epsilon) = (\pi/2)^2 + \frac{1}{2}\epsilon^2 \frac{a}{\sqrt{A_1}}(.279) + \cdots$$

$$= (\pi/2)^2 + .0349 \frac{a}{\sqrt{A_1}} \epsilon^2 + \cdots \quad (92)$$

The magnitude of A_1 is determined by the "power condition"

$$P = q \int_D \varphi(x)\, dx$$

where P is the reactor power level and q is a conversion factor.

Equation (91) implies that there are two equilibrium states for each mode of operation corresponding to the negative and positive values of ϵ. For the fundamental mode this condition may be depicted as shown in Figure 2. For reasonable values of ϵ and (a) the contribution of Γ_m is essentially that given by ψ_m. The impact of the series of cosine terms within the Γ_m function may be seen to be negligible if the peak amplitude $\epsilon A_{13\,MAX}$ of the largest contributor is considered.

$$\epsilon A_{13\,MAX} = \epsilon \frac{2a}{\pi^2} \frac{d_{13}}{1-3^2} \psi_3 \Big|_{x=0}$$

The value of $d_{13} = \frac{.1144}{\sqrt{A_1}}$ was calculated and for a very conservative value of $\epsilon = .1$ and $\frac{a}{\sqrt{A_1}} = .001$.

$$\epsilon A_{13MAX} = .1 \times \frac{2 \times .001}{\pi^2} \times \frac{.1144}{1} \times \frac{1}{8} \times 1$$

$$= 2.91 \times 10^{-6}$$

$$\ll 1 .$$

For each of the equilibrium states there are unique eigenvalues as shown in Figure 3. For a reactor with a negative coefficient of reactivity the fundamental eigenvalue is drawn as a parabola centered at $(\pi/2)^2$ which opens to the right. Thus the equilibrium eigenvalues are always larger than those of the initial state. As has already been deduced for the "T^{-1} model" any decrease in the neutron flux (i.e., negative ϵ) cannot be compatible with an increase in the material buckling. Therefore negative values of ϵ for negative reactivity are unrealistic. For the case of positive reactivity the eigenvalues react in an opposite way as depicted by the dashed line. Values of ϵ which are negative represent possible states of flux, but positive values of ϵ are unrealistic.

When the eigenvalues of the "T^{-1} model" are compared with those of the "$T^{-3/2}$ model" on Figure 3 it is noted that the "T^{-1} model" is affected more by equal perturbations on the system. This result is consistent with that predicted by A. Reichel [3] as can be shown on Figure 1.

There it is observed that the line for the "T^{-1} model" is in very close proximity to Reichel's line of maximum Doppler effect. Hence, one may conclude that the Doppler effect has a more pronounced impact on the "T^{-1} model" reactor than on the "$T^{-3/2}$ model" reactor.

B. STABILITY ANALYSIS OF THE SLAB REACTOR

The stability criterion for the equilibrium states of a slab reactor with negligible concentrations of delayed neutrons is developed as follows. The governing equations, Eqs. (3), (4), and (8) are combined into the equation

$$\frac{\partial \varphi(x,\tau)}{\partial \tau} = \nabla^2 \varphi(x,\tau) + \left[\lambda - a\left(\varphi^{-1/2}(x,\tau) - \varphi_0^{-1/2}(x)\right)\right] \varphi(x,\tau) \quad \text{in } D \tag{93}$$

$$\varphi(x,\tau) = 0 \quad \text{on } \delta D$$

$$\varphi(x,0) = \varphi_0(x).$$

The stability of an equilibrium state is investigated by prescribing an initial deviation, η , in the equilibrium flux and observing the growth of this deviation. If the effect of the deviation increases the flux with time, the system is said to be unstable. If the effect is reversed the system is stable.

The time dependent flux is described by a Taylor series expansion about the equilibrium state, $\varphi(x,\epsilon)$, defined by Eq. (91):

$$\varphi(x,\epsilon,\tau,\eta) = \varphi(x,\epsilon) + \eta\,\varphi_\eta(x,\epsilon,\tau) + \tfrac{1}{2}\eta^2\varphi_{\eta\eta}(x,\epsilon,\tau) + \cdots \tag{94}$$

such that $\lim_{\eta\to 0} \varphi(x,\epsilon,\tau,\eta) = \varphi(x,\epsilon)$.

Equation (93) is differentiated with respect to η and the result is

$$\frac{\partial \varphi_\eta}{\partial \tau} = \nabla^2\varphi_\eta + \left[\lambda(\epsilon) + a\left(\tfrac{1}{2}\varphi^{-1/2} - \varphi_0^{-1/2}(x)\right)\right]\varphi \quad \text{in } D$$

$$\varphi_\eta = 0 \qquad \text{on} \quad \delta D$$

where $\varphi = \varphi(x,\epsilon,\tau,\eta)$.

If the limit as η tends to zero is taken, the equation reduces to

$$\frac{\partial \varphi_\eta}{\partial \tau} = \nabla^2\varphi_\eta + \left[\lambda(\epsilon) - \frac{a}{2}\varphi(x,\epsilon)^{-1/2} - a\,\varphi_0^{-1/2}(x)\right]\varphi_\eta \quad \text{in } D \tag{95}$$

$$\varphi_\eta = 0 \qquad \text{on} \quad \delta D.$$

The solution for $\varphi_\eta(x,\epsilon,\tau)$ is chosen to be

$$\varphi_\eta(x,\epsilon,\tau) = u(x,\epsilon)\exp\left[\alpha(\epsilon)\,\tau\right]. \tag{96}$$

Hence the stability of the system may be analyzed by inquiring whether $\alpha(\epsilon)$ is negative or positive. The value of $\alpha(\epsilon)$ is described by a Taylor series expansion about the parameter ϵ and is given by

$$\alpha(\epsilon) = \alpha_0 + \epsilon\,\dot{\alpha} + \tfrac{1}{2}\epsilon^2\ddot{\alpha} + \cdots. \tag{97}$$

If the proposed solution for $\varphi_\eta(x,\tau,\epsilon)$ (i.e., Eq. (96) is substituted into Eq. (95), then

$$\nabla^2 u(x,\epsilon) + \left[\lambda(\epsilon) - \alpha(\epsilon) + \frac{a}{2}\varphi(x,\epsilon)^{-1/2} - a\,\varphi_0(x)^{-1/2}\right] u(x,\epsilon) = 0 \quad \text{in } D \tag{98}$$

$$u(x,\epsilon) = 0 \quad \text{on } \delta D .$$

Taking the limit as ϵ approaches zero results in the expression:

$$\nabla^2 u(x,0) + \left(\lambda_0 - \alpha_0 - \frac{a}{2}\varphi_0(x)^{-1/2}\right) u(x,0) = 0 \quad \text{in } D \tag{99}$$

$$u(x,0) = 0 \quad \text{on } \delta D .$$

A solution to this equation is offered by a modification to the R. Courant and D. Hilbert perturbation method (provided in Appendix B). Their approximated solution is given by the series

$$u = \bar{u}(x) + \frac{a}{2}\bar{v}(x) + \left(\frac{a}{2}\right)^2 \bar{w}(x) + \cdots \tag{100}$$

Since the contribution of $(a/2)^2$ is very small, only the first and second terms of the series are retained. The technique provides that $\bar{u}(x)$ is the solution to

$$\nabla^2 \bar{u}(x) + (\lambda_{0m} - \alpha_0)\,\bar{u}(x) = 0 \quad \text{in } D$$

$$\bar{u}(x) = 0 \quad \text{on } \delta D ,$$

Since the value of λ_{om} has already been determined in Eq. (72), the only nontrivial solution for $\bar{u}(x)$ is

$$\bar{u}_{m(x)} = u_m \psi_m$$

where ψ_m is defined by Eq. (73). (Proof of this solution is provided in Appendix C). The eigenvalues corresponding to $\bar{u}_{m(x)}$ are $\lambda_{om} = \lambda_{on} - \alpha_o$. Therefore the first term for the series for $\alpha(\epsilon)$ is defined as

$$\alpha_{omm} = \lambda_{on} - \lambda_{om}. \tag{101}$$

The linear term of the series of Eq. (100) is given by

$$\bar{v} = \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \bar{u}_m$$

where $d_{mj} = \int_D \varphi_o^{-1/2}(x) \psi_m \psi_j \, dx$.

The solution for u may readily be seen to be

$$u = u_m \left[\psi_m + \frac{a}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \psi_m + \cdots \right]$$

where $d_{mj} = \int_D \varphi_o^{-1/2}(x) \psi_m \psi_j \, dx.$

This solution may be put into the form afforded by Eq. (81)

$$u = u_m \gamma_m \, . \tag{102}$$

If Eq. (101) is considered, some conclusions as to the stability of the reactor for very small ϵ can be made. For the fundamental mode of the reactor ($m = 1$) the value of α_{01m} is

$$\alpha_{01m} = \lambda_{01} - \lambda_{0m}$$
$$= (\pi/2)^2 (1 - m^2) .$$

The possible values of α_{01m} are

$$\alpha_{01m} = 0 \qquad m = 1 \qquad (102a)$$
$$< -2\pi^2 \qquad m = 3, 5, 7, \cdots .$$

Thus for values of ϵ near zero the first term of the series for $\alpha(\epsilon)$ indicates that the flux is neutrally stable for the fundamental mode of the deviation ($m = 1$) and stable for the higher modes of the deviation ($m > 1$). The question of neutral stability obliges an investigation into the remaining terms of $\alpha(\epsilon)$.

Some conclusive results may be obtained concerning the higher modes of reactor operation. The possible values of $\alpha(\epsilon)$ for the case where $m > 1$ are

$$\alpha(\epsilon) \leq -2\pi^2 \qquad m > n$$
$$= 0 \qquad m = n \qquad (102b)$$
$$\geq 2\pi^2 \qquad m < n .$$

Since $\alpha(\epsilon)$ may be negative, the higher modes are unstable.

The linear term in the series expansion of $\alpha(\epsilon)$ is evaluated by differentiating Eq. (98) with respect to ϵ and thus

$$\nabla^2 \dot{u}(x,\epsilon) + \left[\lambda(\epsilon) - \tfrac{1}{2} a \varphi^{-1/2}(x,\epsilon) - a \varphi_0^{-1/2}(x) - \alpha(\epsilon)\right] \dot{u}(x,\epsilon) + \left[\dot{\lambda}(\epsilon) - \frac{a}{4} \varphi^{-3/2}(x,\epsilon) - \dot{\alpha}(\epsilon)\right] \times u(x,\epsilon) = 0.$$

The value of $\dot{\lambda}(0)$ has already been determined to be zero. If the limit as ϵ approaches zero is taken, then

$$\nabla^2 \dot{u}(x,0) + \left[\lambda_0 - \frac{a}{2} \varphi_0^{-1/2} - \alpha_{om\,m}\right] \dot{u}(x,0) = \left(\frac{a}{4} \varphi_0^{-3/2}(x,0)\, \dot{\varphi}(x,0) + \dot{\alpha}\right) u(x,0). \tag{103}$$

The solvability condition requires that

$$\int_D \left(\frac{a}{4} \varphi_0^{-3/2}(x)\, \dot{\varphi}(x) + \dot{\alpha}\right) u(x)\, \dot{u}^H(x)\, dx = 0 \tag{104}$$

where $\dot{u}^H$ is the homogeneous solution to Eq. (103). If the value of $\alpha_{om\,m}$ is substituted into the homogeneous equation, the result is an equation similar to Eq. (99), thus

$$\dot{u}_m^H = \dot{U}_m^H \Upsilon_m.$$

When u_m and $\dot{u}_m^H$ are used to evaluate the solvability condition the value of $\dot{\alpha}$ is developed. The details of this procedure are developed by using the results for $\varphi_0(x)$, $\dot{\varphi}(x)$, and

$u(x)$ obtained from Eqs. (74), (85), and (102), respectively and $\dot{u}^H(x)$. Substituting these functions into Eq. (104) gives

$$\int_D \left(\frac{a}{4} A_m^{-1/2} \psi_m^{-3/2} \Gamma_m + \dot{\alpha}\right) u_m \Gamma_m \dot{u}_m^H \Gamma_m \, dx = 0$$

or

$$\dot{\alpha} \int_D \Gamma_m^2 \, dx + \frac{a}{4} A_m^{-1/2} \int_D \psi_m^{-3/2} \Gamma_m \Gamma_m^2 \, dx = 0. \quad (105)$$

Consider the coefficient of $\dot{\alpha}$ in view of the definition of Γ_m given by Eq. (81)

$$\int_D \Gamma_m^2 \, dx = \int_D \left[\psi_m + \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{m^2 - j^2} \psi_j\right]^2 dx$$

$$= \int_D \psi_m^2 \, dx + \int_D \frac{2a}{\pi^2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{m^2 - j^2} \psi_j \psi_m \, dx + O(a^2)$$

$$= 1 + 0 + O(a^2)$$

$$\cong 1.$$

An approximation for the linear term of Eq. (105) may similarly be obtained to be:

$$\frac{1}{4} \frac{a}{\sqrt{A_m}} \int_D \psi_m^{-3/2} \Gamma_m \Gamma_m^2 \, dx = \frac{1}{4} \frac{a}{\sqrt{A_m}} \int \psi_m^{-1/2} \psi_m^2 \, dx + O(a^2) \quad (106)$$

if terms of the second power of (a) and larger are considered negligible.

These approximations are substituted into Eq. (105) and

$$\dot{\alpha} \cong -\frac{1}{4}\frac{a}{\sqrt{A_1}}\int_D \psi_n^{-1/2}\psi_m^2\,dx\,. \tag{107}$$

The stability of the system is determined from the evaluation of by substituting the results of Eqs. (101) and (107) into the expansion of Eq. (97). Hence,

$$\alpha(\epsilon) = (\pi/2)^2(n^2-m^2) - \frac{1}{4}\epsilon\frac{a}{\sqrt{A_n}}\int_D \psi_n^{-1/2}\psi_m^2\,dx + \cdots. \tag{108}$$

The additional term enables the further discussion of the stability of the fundamental mode of reactor operation. Thus, for $n = 1$

$$\alpha(\epsilon) = (\pi/2)^2(1-m^2) - \frac{1}{4}\epsilon\frac{a}{\sqrt{A_1}}\int_D \psi_1^{-1/2}\psi_m^2\,dx + \cdots. \tag{109}$$

For all values of $m > 1$ the first term becomes dominant and forces the value of $\alpha(\epsilon)$ to be negative. However, the choice of $m = 1$ defines the value of $\alpha(\epsilon)$,

$$\alpha(\epsilon) = 0 - \frac{1}{4}\epsilon\frac{a}{\sqrt{A_1}}\int_D \psi_1^{3/2}\,dx\,. \tag{110}$$

This integral may be evaluated and the result is,

$$\alpha(\epsilon) = -\epsilon\frac{a}{\sqrt{A_1}}(.279)\,. \tag{111}$$

This result provides the following conclusions;

a) negative reactivity

$$\alpha(\epsilon) < 0 \quad \text{for positive } \epsilon$$

$$\alpha(\epsilon) > 0 \quad \text{for negative } \epsilon$$

These results indicate that the equilibrium solution described by an increase in the flux over the initial flux distribution will be stable. On the other hand, the other equilibrium solution, which was determined to be unacceptable, is unstable.

b) positive reactivity

$$\alpha(\epsilon) > 0 \quad \text{for positive } \epsilon$$

$$\alpha(\epsilon) < 0 \quad \text{for negative } \epsilon$$

The equilibrium state described by a negative value of ϵ is observed to be stable. The other equilibrium state which was concluded to be unrealistic is unstable.

In summary, the stability of a nonlinear reactor was studied by solving the boundary value problem and then considering the effect of an initial deviation on these solutions, called the equilibrium states. The possible equilibrium states are those described by Eq. (91) with a unique eigenvalue for each state given by Eq. (92). However, the

stability criterion predicts that only one of these states are stable and that state will be physically observed in the reactor. For a negative temperature coefficient the stable state is one described by a positive ϵ. For the case of positive temperature coefficient the negative value of ϵ represents the stable condition. Other equilibrium states are unrealistic.

C. EQUILIBRIUM STATES FOR A FINITE CYLINDRICAL REACTOR

The analysis of the equilibrium states of the finite cylindrical reactor can be done in a manner similar to that done for the slab reactor. The equilibrium states and the associated eigenvalues will correspond to those developed for the slab reactor. However differences will be noted because of the differences in the geometry. The flux distribution depends upon a radial coordinate, r, which varies from the center line to the outer radius, r_o, and a longitudinal coordinate, z, which varies from -1 to +1.

The equilibrium solution is developed from the governing equations which are simplified by requiring $\partial \varphi(r,z,\tau)/\partial\tau = 0$.

The governing equations, Eqs. (3) and (8), reduce to the form,

$$\nabla^2 \varphi(r,z) + \left[\lambda + a\left(\varphi^{-1/2}(r,z) - \varphi_o^{-1/2}(r,z)\right)\right]\varphi(r,z) = 0 \quad \text{in } D \tag{112}$$

$$\varphi(r,z) = 0 \quad \text{on } \delta D.$$

The perturbation solution is evaluated by a Taylor series expansion about an initial reactor state, $\varphi_o(r,z)$ and λ_o . If ϵ is the perturbation parameter, which will subsequently be defined, the resultant equilibrium state is

$$\varphi(r,z,\epsilon) = \varphi_o(r,z) + \epsilon \dot{\varphi}(r,z) + \tfrac{1}{2}\epsilon^2 \ddot{\varphi}(r,z) + \cdots \quad (113)$$

$$\lambda(\epsilon) = \lambda_o + \epsilon \dot{\lambda} + \tfrac{1}{2}\epsilon^2 \ddot{\lambda} + \cdots .$$

The initial states, $\varphi_o(r,z)$, of the reqctor is defined as the solution to the system:

$$\nabla^2 \varphi_o(r,z) + \lambda_o \varphi_o(r,z) = 0 \qquad \text{in } D \quad (114)$$

$$\varphi_o(r,z) = 0 \qquad \text{on } \delta D .$$

The function $\varphi_o(r,z)$ is determined in a manner similar to that already provided for Eq. (43). The result is

$$\varphi_o(r,z) = A_m J_o(\mu_m r/r_o) \cos(m\pi/2\, z).$$

If the shape function ψ_m is defined for the cylinder as

$$\psi_m = J_o(\mu_m r/r_o) \cos(m\pi/2\, z) \quad (115)$$

then $$\varphi_o(r,z) = A_m \psi_m . \quad (116)$$

The eigenvalues are also determined by

$$\lambda_o = (\mu_m/r_o)^2 + (m\pi/2)^2 \quad (117)$$

where μ_m is the n^{th} root of the zeroth order Bessel function.

The linear terms of the expansion of Eq. (113) are determined by differentiating Eq. (112) with respect to ϵ and then taking the limit as ϵ tends to zero. The result of these operations is

$$\nabla^2 \dot{\varphi}(r,z) + \left(\lambda_0 - \frac{a}{2}\varphi_0^{-1/2}(r,z)\right)\dot{\varphi}(r,z) = -\dot{\lambda}\,\varphi_0(r,z) \quad \text{in } D \tag{118}$$

$$\dot{\varphi}(r,z) = 0 \quad \text{on } \delta D .$$

In order that a solution is obtained for Eq. (118) the solvability condition (given in Appendix A) requires

$$\int_D -\dot{\lambda}\,\varphi_0(r,z)\,\dot{\varphi}^H(r,z)\,2\pi r\,dr\,dz = 0 \tag{119}$$

where $\dot{\varphi}^H(r,z)$ is the homogeneous solution to Eq. (118). The equation defining $\dot{\varphi}^H(r,z)$, Eq. (120), is solved by a modification to the R. Courant and D. Hilbert perturbation method (provided in Appendix B). Thus,

$$\nabla^2 \dot{\varphi}^H(r,z) + \left(\lambda_0 - \frac{a}{2}\varphi_0^{-1/2}(r,z)\right)\dot{\varphi}^H(r,z) = 0 \quad \text{in } D \tag{120}$$

$$\dot{\varphi}^H(r,z) = 0 \quad \text{on } \delta D .$$

The approximated solution is given by the series

$$\dot{\varphi}^H(r,z) = u_m(r,z) + \frac{a}{2}\sqrt{v_m}(r,z) + O(a^2) . \tag{121}$$

Only the first two terms of the series are retained since the contribution of a^2 is small. The solution for $u_m(r,z)$ is obtained from the equation

$$\nabla^2 u_m(r,z) + \lambda_o u_m(r,z) = 0 \qquad \text{in } D$$

$$u_m(r,z) = 0 \qquad \text{on } \delta D.$$

Since the solution for this equation was already determined for Eq. (114), the result is similar and

$$u_m(r,z) = U_m \psi_m .$$

The value of U_m is determined by the orthonormal condition imposed on the $u_m(r,z)$ function. Thus,

$$\int_D U_m^2 \psi_m^2 \, 2\pi r \, dr \, dz = 1.$$

The integral is evaluated and the result is that

$$U_m^2 = \Theta_m = \frac{1}{\pi r_o^2 J_1^2(\mu_m)} . \tag{122}$$

The second term of the expansion is defined as

$$v_m(r,z) = \sum_{\substack{j=1,3,5\ldots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} u_j$$

where $$d_{mj} = \int_D \varphi_o^{-1/2}(r,z)\, u_m(r,z)\, u_j(r,z)\, 2\pi r\, dr\, dz$$

and $$\lambda_{om} = (\mu_m/r_o)^2 + (m\pi/2)^2 .$$

The value of $v_m(r,z)$ is

$$v_m(r,z) = \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}}\, u_j\, \psi_j$$

where

$$d_{mj} = A_m^{-1/2}\, \Theta_m \Theta_j\, 2\pi \int_D \psi_m^{-1/2}\, \psi_j\, r\, dr\, dz . \quad (123)$$

The solution for $\dot{\varphi}^H(r,z)$ may be determined by evaluating the series of Eq. (121). Thus,

$$\dot{\varphi}^H_{(r,z)} = B_m^H \left[\Theta_m \psi_m + \frac{a}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}}\, \Theta_j\, \psi_j + \cdots \right].$$

If the shape function Υ_m is defined as

$$\Upsilon_m = \Theta_m \psi_m + \frac{a}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}}\, \Theta_j\, \psi_j + \cdots \quad (124)$$

where u_m, λ_{om}, d_{mj}, and ψ_m are defined by Eqs. (122), (117), (123), and (115), the value of $\dot{\varphi}^H(x,z)$ is

$$\dot{\varphi}^{H}_{(r,z)} = B^{H}_{m} \Upsilon_{m} . \tag{125}$$

The values of $\varphi_{0}(r,z)$, given by Eq. (110), and Eq. (125) are used to evaluate the solvability condition, Eq. (119). Thus,

$$\int -\dot{\lambda} A_{m} \Psi_{m} B^{H}_{m} \Upsilon_{m} 2\pi \, dr \, dz = 0$$

or

$$\dot{\lambda} \int \Psi_{m} \Upsilon_{m} r \, dr \, dz = 0 .$$

This equation is satisfied by the condition

$$\dot{\lambda} = 0 . \tag{126}$$

When this result is substituted into Eq. (118) the solution for $\dot{\varphi}_{(r,z)}$ is the homogeneous solution to Eq. (118) or the solution to Eq. (120). Thus the value of $\dot{\varphi}(r,z)$ is

$$\dot{\varphi}(r,z) = B_{m} \Upsilon_{m} . \tag{127}$$

The value of the perturbation parameter, ϵ, is chosen to represent the average magnitude of the linear term in the expansion of Eq. (113) weighted by the Υ_{m} function and measured in units of the maximum initial flux. Thus

$$\epsilon = \frac{1}{A_{m}} \int_{D} \Upsilon_{m} \left[\varphi(\chi,\epsilon) - \varphi_{0}(\chi) \right] d\chi . \tag{128}$$

If Eq. (128) is differentiated with respect to ϵ and the limit is taken as ϵ approaches zero, the result is

$$\frac{A_m}{B_m} = \int_D \Upsilon_m^2 \, 2\pi r \, dr \, dz$$

$$= \int_D \left[\Theta_m \psi_m + \frac{a}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \Theta_j \psi_j \right]^2 2\pi r \, dr \, dz .$$

If the quantity within the bracket is expanded and terms of a^2 are considered small, then

$$\frac{A_m}{B_m} = \int_D \Theta_m^2 \psi_m^2 \, 2\pi r \, dr \, dz$$

$$+ a \int_D \Theta_m \psi_m \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \Theta_j \psi_j \, 2\pi r \, dr \, dz$$

$$+ O(a^2).$$

Because of the orthogonality of the ψ_m function the second term vanishes and

$$\frac{A_m}{B_m} = \int \Theta_m^2 \psi_m^2 \, 2\pi r \, dr \, dz$$

$$= 1.$$

Thus, $A_m = B_m$ and Eq. (127) may be rewritten as

$$\dot{\varphi}(r,z) = A_m \Upsilon_m . \tag{129}$$

The quadratic terms of the expansion for $\varphi(r,z,\epsilon)$ and $\lambda(\epsilon)$ given by Eq. (113) are evaluated by differentiating Eq. (112) twice with respect to ϵ and then let ϵ tend to zero. The result is

$$\nabla^2 \ddot{\varphi}(r,z) + \left(\lambda_0 - \frac{a}{2} \varphi_0^{-1/2}(r,z)\right) \ddot{\varphi}(r,z) = \tag{130}$$

$$-\ddot{\lambda} \varphi_0(r,z) + \frac{a}{4} \varphi_0^{-3/2}(r,z) \dot{\varphi}^2(r,z) \quad \text{in } D$$

$$\ddot{\varphi}(r,z) = 0 \qquad \text{on } \delta D .$$

The solvability condition requires that

$$\int_D \left(-\ddot{\lambda} \varphi_0(r,z) + \frac{a}{4} \varphi_0^{-3/2}(r,z) \dot{\varphi}^2(r,z)\right) \ddot{\varphi}^H(r,z)\, 2\pi r\, dr\, dz = 0 \tag{131}$$

where $\ddot{\varphi}^H(r,z)$ is the homogeneous solution to Eq. (130). This was previously determined for Eq. (120) and from that result

$$\ddot{\varphi}^H(r,z) = C_m^H \Upsilon_m . \tag{132}$$

The value of $\ddot{\lambda}$ may be determined by substituting the correct functions determined above. Thus,

$$\ddot{\lambda} \int_D A_m \psi_m C_m^H \Upsilon_m \, 2\pi r \, dr \, dz =$$

$$\frac{a}{4} \int_D A_m^{1/2} \psi_m^{-3/2} \Upsilon_m C_m^H \Upsilon_m \, 2\pi r \, dr \, dz .$$

The coefficient of $\ddot{\lambda}$ is evaluated as follows:

$$A_m \int_D \psi_m \Upsilon_m \, 2\pi r \, dr \, dz$$

$$\cong A_m \int_D \psi_m \left[\textcircled{H} \psi_m + \frac{a}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \textcircled{H}_j \psi_j \right] \times 2\pi r \, dr \, dz$$

$$\cong \frac{A_m}{U_m}$$

where U_m is defined by Eq. (122). The integral on the right hand side is evaluated as follows:

$$\frac{a}{4} A_m^{1/2} \int_D \psi_m^{-3/2} \Upsilon_m^3 \, 2\pi r \, dr \, dz$$

$$\cong \frac{a}{4} A_m^{1/2} \int_D \psi_m^{3/2} \textcircled{H}_m^3 \, 2\pi r \, dr \, dz$$

if terms multiplied by a^2 are considered small. The value of $\ddot{\lambda}$ is

$$\ddot{\lambda} = \frac{a}{4} \frac{1}{\sqrt{A_m}} \int_D \psi_m^{3/2} \textcircled{H}_m^4 \, 2\pi r \, dr \, dz . \qquad (133)$$

For the fundamental mode ($m = 1$)

$$\ddot{\lambda} = \frac{.563}{r_o^2} \frac{a}{\sqrt{A_1}} \quad . \tag{134}$$

The various terms of the expansion of Eq. (113) may be combined to give the equilibrium states of the reactor. Thus, substituting Eqs. (116) and (129) into the expansion for $\varphi(r, z, \epsilon)$ gives

$$\varphi(r, z, \epsilon) = A_m \left[\psi_m + \epsilon \Upsilon_m + O(\epsilon^2) \right] \tag{135}$$

and substituting Eqs. (117), (126), and (133) into the expansion for $\lambda(\epsilon)$ gives

$$\lambda(\epsilon) = (\mu_m / r_o)^2 + (m\pi/2)^2 + \frac{1}{2} \epsilon^2 \frac{a}{\sqrt{A_m}} \frac{1}{4} \int_D \psi_m^{3/4} \Theta_m^4 \, 2\pi r \, dr \, dz + \cdots .$$

For the fundamental mode of the reactor ($m = 1$) the integral is evaluated and

$$\lambda(\epsilon) = (\mu_1 / r_o)^2 + (\pi/2)^2 + \epsilon^2 \frac{.563}{r_o^2} \frac{a}{\sqrt{A_1}} + \cdots . \tag{136}$$

It is apparent that for any mode of reactor operation there will be two equilibrium states defined by Eq. (135) and each of these states will have unique eigenvalues defined by Eq. (136). However, only one of these values will represent an acceptable reactor state.

D. STABILITY ANALYSIS OF THE FINITE CYLINDRICAL REACTOR

The stability criterion for a finite cylindrical reactor with negligible fraction of delayed neutrons is determined by the methods previously considered. An initial deviation, η , due to temperature feedback is introduced into the equilibrium states of the reactor and the temporal effect is noted.

The governing equation has the form given by Eq. (93) where the functional dependence of the variable is changed to accommodate the cylindrical geometry

$$\frac{\partial \varphi(r,z,\tau)}{\partial \tau} = \nabla^2 \varphi(r,z,\tau) + \left[\lambda - a\left(\varphi^{-1/2}_{(r,z,\tau)} - \varphi_0(r,z)\right)\right]\varphi(r,z,\tau) \quad \text{in } D \tag{137}$$

$$\varphi(r,z,\tau) = 0 \qquad \text{on } \delta D$$

$$\varphi(r,z,0) = \varphi_0(r,z) .$$

The deviation is described by a Taylor series expansion about the equilibrium states, Eq. (135), given by

$$\varphi(r,z,\epsilon,\tau,\eta) = \varphi(r,z,\epsilon) + \eta\,\varphi_\eta(r,z,\epsilon,\tau) + \tfrac{1}{2}\eta^2\varphi_{\eta\eta}(r,z,\epsilon,\tau) + \cdots \tag{138}$$

such that

$$\lim_{\eta \to 0} \varphi(r,z,\epsilon,\tau,\eta) = \varphi(r,z,\epsilon) .$$

Eq. (137) is differentiated with respect to η and then the limit is taken as the value of η tends to zero. The result of these operations yields the equation

$$\partial \varphi_\eta(r,z,\epsilon,\tau) = \nabla^2 \varphi_\eta(r,z,\epsilon,\tau) + \left[\lambda(\epsilon) - \frac{a}{2}\varphi^{-1/2}(r,z,\epsilon,\tau) - a\,\varphi_0^{-1/2}(r,z)\right]\varphi_\eta(r,z,\epsilon,\tau) \quad \text{in } D \tag{139}$$

$$\varphi_\eta(r,z,\epsilon,\tau) = 0 \qquad \text{on } \delta D.$$

A form for the solution of $\varphi_\eta(r,z,\epsilon\tau)$ is chosen to be

$$\varphi_\eta(r,z,\epsilon,\tau) = u(r,z,\epsilon)\,\exp\left[\alpha(\epsilon)\,\tau\right]. \tag{140}$$

The stability of the system may be determined by investigating the sign of $\alpha(\epsilon)$. If the sign of $\alpha(\epsilon)$ were positive, then the exponential would increase with time. Hence, the contribution of φ_η in the series of Eq. (138) would be unbounded with time and the system would be unstable. On the other hand, negative value of $\alpha(\epsilon)$ would indicate that the φ_η contribution would die and stability would be assured. The value of $\alpha(\epsilon)$ will be determined by a series expansion given by

$$\alpha(\epsilon) = \alpha_0 + \epsilon\dot{\alpha} + \tfrac{1}{2}\epsilon^2\ddot{\alpha} + \cdots. \tag{141}$$

The following will be directed toward evaluating each of the terms of this expansion.

If the proposed solution for $\varphi_\eta(r,z,\tau,\epsilon)$, Eq. (140), is substituted intoEq. (139) the result is,

$$\nabla^2 u + \left[\lambda(\epsilon) - \alpha(\epsilon) + \frac{a}{2}\varphi^{-1/2}(r,z,\epsilon) - a\,\varphi_o^{-1/2}(r,z)\right] u = 0 \quad \text{in } D \tag{142}$$

$$u(r,z,\epsilon) = 0 \quad \text{on } \delta D$$

where $u = u(r,z,\epsilon)$. The first term of the series of $\alpha(\epsilon)$ is evaluated by taking the limit of Eq. (142) as ϵ approaches zero. Thus,

$$\nabla^2 u(r,z) + \left[\lambda_{om} - \alpha_o - \tfrac{1}{2} a\,\varphi_o^{-1/2}(r,z)\right] \times u(r,z) = 0 \quad \text{in } D \tag{143}$$

$$u(r,z) = 0 \quad \text{on} \quad \delta D .$$

The solution to this equation may be determined by the approximation given by the R. Courant and D. Hilbert perturbation technique. A solution to Eq. (143) is developed about the series

$$u(r,z) = \bar{u}(r,z) + \frac{a}{2}\bar{N}(r,z) + O(a^2).$$

The solution for $\bar{u}(r,z)$ is determined about the solution to the equation

$$\nabla^2 \bar{u}(r,z) + (\lambda_{om} - \alpha_o)\,\bar{u}(r,z) = 0 \quad \text{in } D$$

$$\bar{u}(r,z) = 0 \quad \text{on} \quad \delta D .$$

It may be noted that the solution to this equation is similar to that observed in Eq. (114). However, by the results of Appendix C one may conclude that the solution for $\bar{u}(r,z)$ is given by $\bar{u}(r,z) = U_m \psi_m$ where the function ψ_m is defined in Eq. (115) and $\lambda_{om} = \lambda_{on} - \alpha_o$. Thus,

$$\alpha_o = \lambda_{on} - \lambda_{om} \tag{144}$$

and the linear term of the $\alpha(\epsilon)$ series of Eq. (141) is defined. (A discussion of α_o will subsequently be made.) Since the $\bar{u}(r,z)$ function must be orthonormal the constant $\bar{U}_m$ may be evaluated by the normality condition

$$\int_D \bar{U}_m^2 \psi_m^2 \, 2\pi r \, dr \, dz = 1 .$$

The value of $\bar{U}_m^2$ is

$$\bar{U}_m^2 = \Theta_m^2 = \frac{1}{\pi r_o^2 J_1^2(\mu_m)} \tag{122}$$

which has previously been determined by Eq. (122). Thus $\bar{u}(r,z)$ is

$$\bar{u}(r,z) = \Theta_m \psi_m .$$

The second term of the expansion $u(r,z)$ is defined as

$$\bar{v}(r,z) = \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \bar{u}_j(r,z)$$

or

$$\bar{v}_m(r,z) = \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \textcircled{H}_j \psi_j$$

where d_{mj} is defined by Eq. (123). The solution for the $u(r,z)$ function from Eq. (143) is determined as

$$\begin{aligned} u(r,z) &= U_m \left[\bar{u}(r,z) + \frac{\epsilon}{2} \bar{v}(r,z) + \cdots \right] \\ &= U_m \left[\textcircled{H}_m \psi_m + \frac{\epsilon}{2} \sum_{\substack{j=1,3,5,\cdots \\ \neq m}}^{\infty} \frac{d_{mj}}{\lambda_{om} - \lambda_{oj}} \textcircled{H}_j \psi_j + \cdots \right] \\ &= U_m \Upsilon_m \end{aligned} \tag{145}$$

where Υ_m is defined by Eq. (124).

The value of α_{omm} as evaluated from Eq. (144) is rewritten below

$$\alpha_{omm} = \lambda_{om} - \lambda_{om}. \tag{146}$$

If the value of λ_o as given by Eq. (117) are used, the result is

$$\alpha_{omm} = \left(\frac{1}{r_o}\right)^2 \left(\mu_m^2 - \mu_m^2\right) + \left(\pi/2\right)^2 \left(m^2 - m^2\right). \tag{146a}$$

For the fundamental mode ($m = 1$) this equation reduces to the form

$$\alpha_{0lm} = \left(\frac{1}{r_o}\right)^2 \left(\overline{2.4048}^2 - \mu_m^2\right) + \left(\pi/2\right)^2 \left(1 - m^2\right).$$

For values of ϵ near zero the values of α_{0lm} are

$$\alpha_{0lm} = 0 \qquad m = n$$
$$< -2\pi^2 \qquad m > n \quad .$$

Thus for all modes of the deviation except the fundamental the sign of α_{0nm} is negative and any initial deviation of the system will decay with time. However, for the value $m = n$ the first term evaluation of the stability indicates that there is a neutral stability for the fundamental mode. Thus it is necessary to investigate the higher order terms of the expansion of $\alpha(\epsilon)$. For the higher modes of reactor operation $\alpha(\epsilon)$ may have the following values,

$$\alpha(\epsilon) > 2\pi^2 \qquad m < n$$
$$= 0 \qquad m = n$$
$$< -2\pi^2 \qquad m > n .$$

Thus the value of $\alpha(\epsilon)$ may be positive when $m < n$ for $n > 1$. Since the value of α_{0nm} is so large when compared to the linear term (order of ϵ) of $\alpha(\epsilon)$, one may conclude that the first term of the series is dominant and the higher modes are unstable.

The linear term in the expansion of $\alpha(\epsilon)$, Eq. (141), may be evaluated by differentiating Eq. (142) with respect to ϵ and taking the limit as ϵ tends to zero. Thus,

$$\nabla^2 \dot{u}(r,z) + \left[\lambda_0 - \tfrac{1}{2} a\, \varphi_0^{-1/2}(r,z) - \alpha_{omm}\right] \dot{u}(r,z) = \left(\tfrac{1}{4} a\, \varphi_0^{-3/2}(r,z)\, \dot{\varphi}(r,z) + \dot{\alpha}\right) u(r,z) \quad \text{in } D \tag{147}$$

$$\dot{u}(r,z) = 0 \qquad \text{on } \delta D .$$

The solvability condition requires that

$$\int_D \left(\dot{\alpha} + \tfrac{1}{4} a\, \varphi_0^{-3/2}(r,z)\, \dot{\varphi}(r,z)\right) u(r,z)\, \dot{u}^H(r,z)\, 2\pi r\, dr\, dz = 0$$

where $\dot{u}^H(r,z)$ is the homogeneous solution to Eq. (147). This solution was previously developed for Eq. (143) and the solution is

$$\dot{u}^H(r,z) = \dot{u}_m \Gamma_m .$$

With the solution for $\dot{u}^H(r,z)$ and the folutions for $u(r,z)$, $\varphi_0(r,z)$, and $\dot{\varphi}(r,z)$ given by Eqs. (145), (116), and (129) the value of $\dot{\alpha}$ may be evaluated from the solvability condition. Thus,

$$\dot{\alpha}\, u_m \dot{u}_m \int_D \Upsilon_m^2 \, 2\pi r \, dr \, dz =$$
$$-\frac{a}{4} A_m^{-1/2} u_m \dot{u}_m \int_D \psi_m^{-3/2} \Upsilon_m \Upsilon_m^2 \, 2\pi r \, dr \, dz$$

The coefficient of $\dot{\alpha}$ was evaluated for Eq. (128). The result indicated that if the terms of order a^2 are considered very small, the integral was equal to one. The value of $\dot{\alpha}$ is

$$\dot{\alpha} = \frac{-1}{4} \frac{a}{\sqrt{A_m}} \int_D \psi_m^{-3/2} \Upsilon_m \Upsilon_m^2 \, 2\pi r \, dr \, dz . \qquad (148)$$

Since the fundamental mode of the reactor is of greatest interest, the integral of Eq. (148) is evaluated for $m = n = 1$. Thus,

$$\dot{\alpha} = \frac{-1}{4} \frac{a}{\sqrt{A_1}} \int_D \psi_1^{-3/2} \Upsilon_1^3 \, 2\pi r \, dr \, dz$$
$$\cong \frac{-1}{4} \frac{a}{\sqrt{A_1}} \int_D \Theta_1^3 \psi_1^{3/2} \, 2\pi r \, dr \, dz$$
$$\cong -\frac{3.73}{r_o} \frac{a}{\sqrt{A_1}} .$$

The results of Eqs. (146) and (148) may be combined into the expansion of $\alpha(\epsilon)$, Eq. (141),

$$\alpha(\epsilon) = \lambda_{om} - \lambda_{om} - \epsilon \frac{1}{4} \frac{a}{\sqrt{A_m}} \int_D \psi_n^{-3/2} \Upsilon_n \Upsilon_m^2 \, 2\pi r \, dr \, dz \tag{149}$$

and for the fundamental mode

$$\alpha(\epsilon) = \left(\frac{1}{r_o}\right)^2 \left(\overline{2.4048}^2 - \mu_m^2\right) + \left(\pi/2\right)^2 (1 - m^2) - \epsilon \frac{3.73}{r_o} \frac{a}{\sqrt{A_1}} + O(\epsilon^2). \tag{150}$$

From the previous analysis values of $m > 1$ have been found to favor stability. For the value of $m = 1$ the following becomes evident,

$$\alpha(\epsilon) = -\epsilon \frac{3.73}{r_o} \frac{a}{\sqrt{A_1}} + O(\epsilon^2). \tag{151}$$

For the case of negative reactivity the sign of $\alpha(\epsilon)$ will be determined by the value of ϵ. A positive value of ϵ will permit a negative sign of $\alpha(\epsilon)$. The deviation associated with $\alpha(\epsilon)$ will decrease with time and stability results. For positive reactivity the sign of (a) is changed and the negative value of ϵ permits stability. The other states of reactor equilibrium are unrealistic as determined by previous arguments.

APPENDIX A

THE SOLVABILITY CONDITION FOR NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS

Given the equation

$$\nabla^2 \varphi(x) + \lambda \varphi(x) = f(x) \qquad \text{in } D \tag{A-1}$$

$$\varphi(x) = 0 \qquad \text{on } \delta D ,$$

prove that this equation has a solution if and only if the function $f(x)$ satisfies the condition:

$$\int_D f(x)\, \varphi^H(x)\, dV = 0$$

where

$$\nabla^2 \varphi^H(x) + \lambda \varphi^H(x) = 0 \qquad \text{in } D \tag{A-2}$$

$$\varphi^H(x) = 0 \qquad \text{on } \delta D .$$

Multiplying Eq. (A-1) by $\varphi^H(x)$ and integrating throughout the domain the following results:

$$\int_D \left(\varphi^H(x)\, \nabla^2 \varphi(x) + \lambda \varphi^H(x)\, \varphi(x) \right) dV = \int_D \varphi^H(x) \cdot f(x)\, dV . \tag{A-3}$$

From Eq. (A-2) $\nabla^2 \varphi^H_{(x)} = -\lambda \varphi^H_{(x)}$ and this is substituted into Eq. (A-3). Thus,

$$\int_D \left(\varphi^H_{(x)} \nabla^2 \varphi_{(x)} - \varphi_{(x)} \nabla^2 \varphi^H_{(x)}\right) dV = \int_D \varphi^H_{(x)} f_{(x)} \, dV . \tag{A-4}$$

By Green's Theorem the left hand side of Eq. (A-4) may be transformed into a surface integral:

$$\int_S n \left(\varphi^H_{(x)} \nabla \varphi_{(x)} - \varphi_{(x)} \nabla \varphi^H_{(x)}\right) dS = \int_D f_{(x)} \varphi^H_{(x)} \, dV .$$

Thus, if $\varphi_{(x)}$ is to be the solution to Eq. (A-1), the homogeneous boundary condition on δD must be satisfied. Therefore,

$$\int_D f_{(x)} \varphi^H_{(x)} \, dV = 0 \tag{A-5}$$

Q.E.D.

APPENDIX B

A PERTURBATION METHOD FOR LINEAR EIGENVALUE PROBLEMS [7]

The solutions to equations of the form

$$\nabla^2 \bar{u}_m - \epsilon r(x) \bar{u}_m + \lambda_m \bar{u}_m = 0 \quad \text{in } D \tag{B-1}$$

$$\bar{u}_m = 0 \quad \text{on } \delta D ,$$

where $\epsilon\, r(x)$ represents a small perturbation, $(|\epsilon r(x)| \ll \lambda_m)$, are provided by a series expansion about the solution of the equation

$$\nabla^2 u_m + \lambda_m u_m = 0 \quad \text{in } D \tag{B-2}$$

$$u_m = 0 \quad \text{on } \delta D .$$

The eigenvalues are held constant while the eigenfunctions are perturbed in accordance with the Taylor series given by

$$\bar{u}_m = u_m + \epsilon v_m + \tfrac{1}{2}\epsilon^2 w_m + \cdots . \tag{B-3}$$

If the series is substituted into Eq. (B-1), the result is

$$\begin{aligned} &\nabla^2 (u_m + \epsilon v_m + \tfrac{1}{2}\epsilon^2 w_m + \cdots) \\ &\quad - \epsilon r(x) (u_m + \epsilon v_m + \tfrac{1}{2}\epsilon^2 w_m + \cdots) \\ &\quad + \lambda_m (u_m + \epsilon v_m + \tfrac{1}{2}\epsilon^2 w_m + \cdots) \\ &\quad = 0. \end{aligned}$$

By separating terms of the same power in ϵ, the following equations are obtained:

$$\nabla^2 u_m + \lambda_m u_m = 0 \tag{B-4}$$

$$\nabla^2 v_m + \lambda_m v_m = r(x)\, u_m \tag{B-5}$$

$$\nabla^2 w_m + \lambda_m w_m = r(x)\, u_m . \tag{B-6}$$

Multiply Eq. (B-5) by u_ℓ and integrate over the volume, dV :

$$\int_D (u_\ell \nabla^2 v_m + \lambda_m u_\ell v_m)\, dV = \int_D r(x)\, u_m u_\ell \, dV$$

$$\int_D (u_\ell \nabla^2 v_m - v_m \nabla^2 u_\ell + v_m \nabla^2 u_\ell)\, dV + \int_D \lambda_m u_\ell v_m \, dV = \int_D r(x)\, u_m u_\ell \, dV . \tag{B-7}$$

By Green's Theorem the following volume integral may be transformed into a surface integral

$$\int_D (\nabla^2 v_m - v_m \nabla^2 u_\ell)\, dV = \int_S n (u_\ell \nabla v_m - v_m \nabla u_\ell)\, dS .$$

However, the boundary conditions specify that u_ℓ and v_m are everywhere zero on the surface and the surface integral vanishes.

Equation (B-7) may be rewritten as

$$\int_D (v_m \nabla^2 u_\ell + \lambda_m u_\ell v_m)\, dV = d_{m\ell}$$

where

$$d_{m\ell} = \int_D r(x)\, u_m u_\ell\, dV. \tag{B-8}$$

Equation (B-2) reveals

$$\nabla^2 u_\ell = -\lambda_\ell u_\ell$$

which is substituted into Eq. (B-8) to give

$$(\lambda_m - \lambda_\ell)\, a_{m\ell} = d_{m\ell} \tag{B-9}$$

where $$a_{m\ell} = \int_D v_m u_\ell\, dV .$$

The value of the integral a_{mm} is evaluated if the normalizing condition $\int_D u_m^2\, dV = 1$ is imposed. It is clear that

$$\int_D \left(u_m + \epsilon v_m + \tfrac{1}{2}\epsilon^2 w_m + \cdots\right)^2 dV = 1. \tag{B-10}$$

By equating powers of ϵ on the right and left sides of Eq. (B-10)

$$\int u_m^2\, dx = 1$$

$$\int u_m v_m\, dx = 0$$

or $$\epsilon\, a_{mm} = 0 .$$

The function $\nu_m(x)$ is determined by a series of orthogonal functions given by

$$\nu_m(x) = \sum_{j=1}^{\infty} b_{mj}\, u_j \quad . \tag{B-11}$$

Multiplying both sides by u_ℓ and integrating over the domain

$$\int_D \nu_m u_\ell \, dV = \int_D \sum_{j=1}^{\infty} b_{mj} u_j u_\ell \, dV .$$

Assuming that the expansion exists and that the integral of the infinite series is equal to the sum of the integrals,

$$a_{m\ell} = \sum_{\substack{j=1 \\ \neq m}}^{\infty} b_{mj} \int_D u_j u_\ell \, dV$$

$$a_{mm} = 0 .$$

Since $$\int_D u_j u_\ell \, dV = 0 \qquad j \neq \ell$$

and $$\int_D u_j u_\ell \, dV = 1 \qquad j = \ell ,$$

the constants b_{mj} are evaluated as $b_{mj} = a_{mj}$.

However, from Eq. (B-9) it is obvious that $b_{nj} = \frac{d_{nj}}{\lambda_n - \lambda_j}$.

Thus the series of orthogonal function given by Eq. (B-11) may be evaluated as

$$v_n(x) = \sum_{\substack{j=1,2,3\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{\lambda_n - \lambda_j} u_j(x) .$$

The series of Eq. (B-3) may be evaluated as

$$\bar{u}_n(x) = u_n(x) + \epsilon \sum_{\substack{j=1,2,3\cdots \\ \neq n}}^{\infty} \frac{d_{nj}}{\lambda_n - \lambda_j} u_j(x) + \cdots$$

where $d_{nj} = \int_D r(x)\, u_n u_j \, dV .$

APPENDIX C

Given the equation

$$\nabla^2 \psi_m(x) + \lambda_m \psi_m(x) = 0 \qquad \text{in } D \tag{C-1}$$

$$\psi_m(x) = 0 \qquad \text{on } \delta D$$

with eigenfunctions $\psi_m(x)$ and eigenvalues λ_m. Prove that if

$$\nabla^2 \varphi(x) + \lambda \varphi(x) = 0 \qquad \text{in } D \tag{C-2}$$

$$\varphi(x) = 0 \qquad \text{on } \delta D,$$

where $\lambda \neq \lambda_m$ is a new physical parameter, then the nontrivial solutions to Eq. (C-2) are

$$\varphi(x) = \psi_m(x) \qquad \lambda = \lambda_m$$

where m and n are independent subscripts.

Multiply Eq. (C-2) by $\psi_n(x)$ and integrate throughout the domain.

$$\int_D \psi_n(x) \nabla^2 \varphi(x)\, dV + \lambda \int_D \psi_n(x) \varphi(x)\, dV = 0$$

By adding and subtracting $\varphi(x)\,\nabla^2\psi_m(x)$

$$\int_D \left(\psi_m(x)\,\nabla^2\varphi(x) - \varphi(x)\,\nabla^2\psi_m(x)\right) dV \qquad \text{(C-3)}$$

$$\int_D \varphi(x)\,\nabla^2\psi_m(x)\,dV + \lambda\int_D \psi_m(x)\,\varphi(x)\,dV = 0 .$$

From Green's Theorem the first integral may be transformed into a surface integral by

$$\int_D \left(\psi_m(x)\,\nabla^2\varphi(x) - \varphi(x)\,\nabla^2\psi_m(x)\right) dV$$

$$= \int_S n\left(\psi_m(x)\,\nabla\varphi(x) - \varphi(x)\,\nabla\psi_m(x)\right) dS .$$

However, from the homogeneous boundary conditions of $\psi_m(x) = \varphi(x) = 0$ on δD the surface integral vanishes. Thus Eq. (C-3) is

$$\int_D \varphi(x)\,\nabla^2\psi_m(x)\,dV + \lambda\int \psi_m(x)\,\varphi(x)\,dV = 0. \qquad \text{(C-4)}$$

From Eq. (C-1) $\nabla^2\psi_m(x) = -\lambda_m\psi_m(x)$;

therefore

$$-\lambda_m\int_D \varphi(x)\,\psi_m(x)\,dV + \lambda\int_D \psi_m(x)\,\varphi(x)\,dV = 0$$

or $$(\lambda - \lambda_m)\int_D \varphi(x)\,\psi_m(x)\,dV = 0 .$$

If $(\lambda - \lambda_m) \neq 0$, then the integral must vanish throughout the domain and $\varphi(x)$ is orthogonal to $\psi_m(x)$. Hence,

$$\varphi(x) = \psi_m(x)$$

and the eigenvalues of Eq. (C-2) (i.e., λ) are

$$\lambda = \lambda_m .$$

Q.E.D.

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13. ABSTRACT

The analysis of nonlinear reactor kinetics with Doppler feedback is developed by the use of the Poincaré-Lindstedt perturbation technique. The two types of temperature dependent reactivity coefficients to be investigated are

$$\frac{dk}{dT} \propto \frac{1}{T} \quad \text{and} \quad \frac{dk}{dT} \propto \frac{1}{T^{3/2}}.$$

Each of these reactivity models are analyzed for the reactor with positive and negative coefficients of reactivity and infinite slab and finite cylindrical geometries. The Poincaré-Lindstedt perturbation solution is constructed in the form of an expansion about the unperturbed state of the reactor. Each of the perturbed equilibrium solutions is subsequently analyzed for stability.

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